SIMPLE ANALOG COMPUTER OSCILLOSCOPE DISPLAYS

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1. INTRODUCTION

An effective oscilloscope display can provide a great deal of insight into the solution of a problem in dynamics. Spear[1] reports that a picture of the collision of two bodies in space was beneficial in determining the limitations of the equations which were involved in his study. Others have had similar experiences,[2,3] but a general treatment of this subject has been neglected in the literature, perhaps because such displays were not possible with general-purpose analog computing equipment until recently. Modern analog computers contain the means to program a simple computer display easily and quickly.

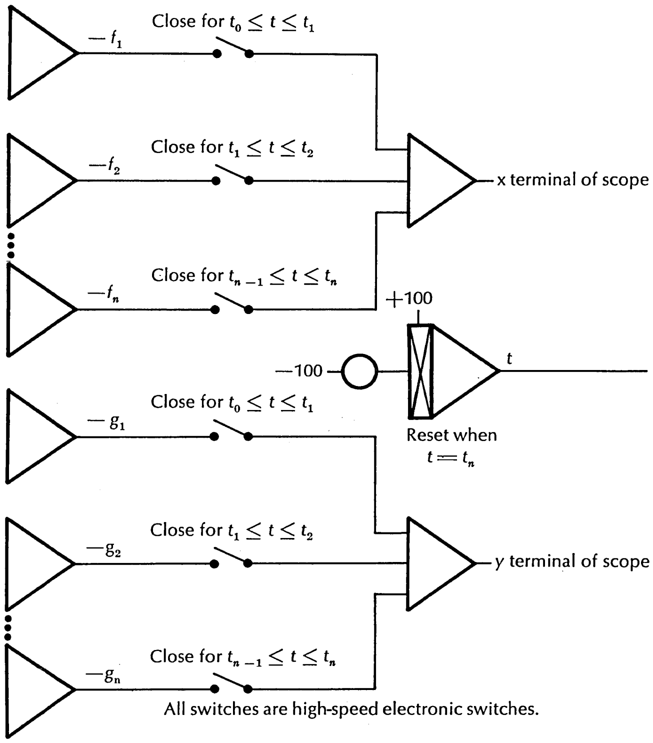
This paper attempts to generalize and formalize some methods which can be used to generate effective two-dimensional displays for dynamic systems.

The basic equipment requirement is a multi-speed general-purpose analog computer with electronic mode control for the integrators, and high-speed electronic switches. Such equipment is commercially available today.

2. THE BASIC METHOD

Figure 1 shows an obvious method of tracing a picture on the oscilloscope. When plotted against one another, the waveforms $f_i(t)$ and $g_i(t)$ produce a *continuous* graph or figure on the scope, which will be referred to as (f_i, g_i). The final picture (x, y) may consist of many figures, and switching is necessitated whenever disjoint figures are desired. Figure 2 is a switching diagram more easily realizable, considering the nature of electronic switches. All switching at this stage would be unnecessary if a multi-channel scope with independent x and y plates for each channel were available. Such an instrument would also take advantage of the parallel nature of analog computing equipment. Unfortunately, scopes with more than two independent channels are not available as yet.

The entire process of switching all pairs of waveforms (f_i, g_i) onto the x and y plates of a scope occurs within a period of 50 msec or less, in order to produce at least 20 pictures per second. At this rate, a



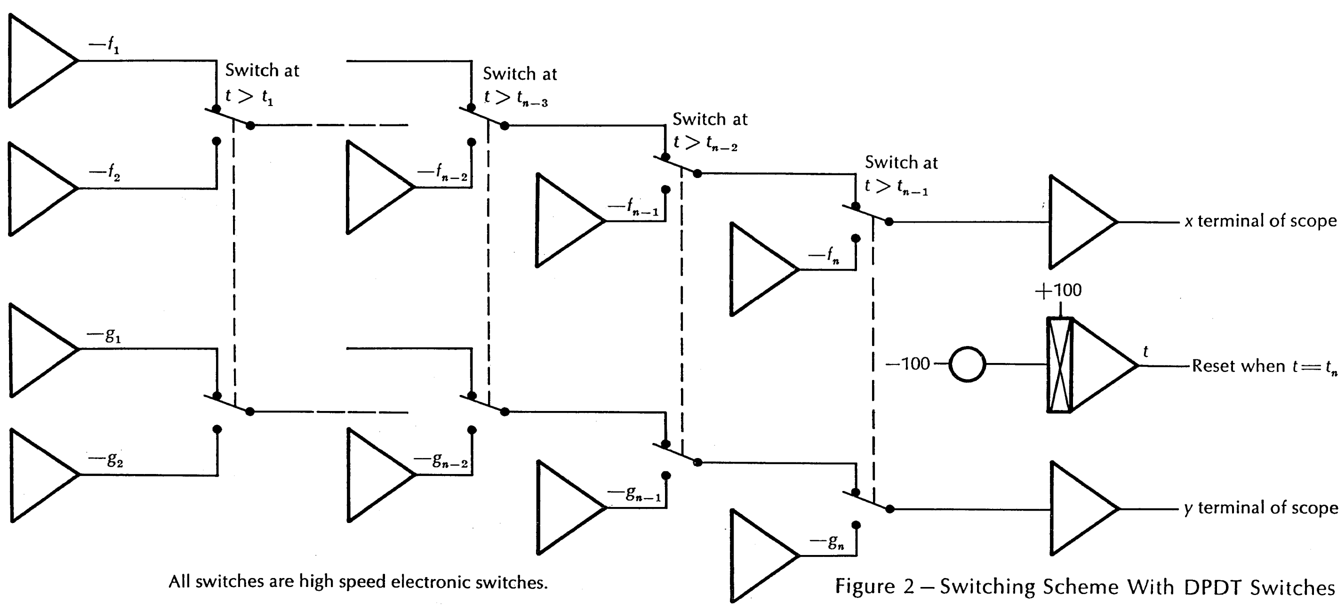
All switches are high-speed electronic switches.

Figure 1 — Switching Scheme

sequence of pictures fools the eye and appears as continuous motion. A phosphor with low persistance should be used for the scope screen to prevent successive rapidly-changing pictures from appearing simultaneously.

At least one figure in the picture is changing its position in the sequence because the non-repetitive portion of the computer is solving a problem, the variables of which are used as parameters in generating the figures. Only linear or angular problem coordinates will be considered here, so that it is only necessary to consider rotations and translations of figures if all motion takes place in a plane. Some three-dimensional effects can be obtained by introducing perspective, shrinking figures as they recede in the distance, etc., but such effects are beyond the scope of this paper.

There are actually more basic figures than (f_i, g_i). Consider, for example, projecting an ellipse on the screen of an oscilloscope. An oscillator circuit can be used to generate waveforms which will draw an ellipse on the screen, but the center of the ellipse will be at the origin and the major and minor axes will coincide with the *x*- and *y*-axes. The ellipse may be rotated and translated during the computation. Let



All switches are high speed electronic switches.

Figure 2 — Switching Scheme With DPDT Switches

the basic untransformed figures be denoted by (F_i, G_i). These figures may be acted upon by the transformation equations:

$$f_i = X_t + F_i \cos\theta - G_i \sin\theta \quad (1)$$

$$g_i = Y_t + F_i \sin\theta + G_i \cos\theta \quad (2)$$

to properly locate and orient (F_i, G_i) on the screen. The linear variables X_t and Y_t and the rotation angle θ (counter-clockwise) are functions of problem variables and are essentially constant within each repetitive cycle, and may be treated as such. The rotation is about the point (0, 0).

The big problem for most computer displays lies in generating (F_i, G_i). A great deal of ingenuity, intuition, and imagination can be exercised in this area, with basic waveforms being generated in a variety of ways. The rest of this paper will be concerned with generating (F_i, G_i) by an integration process. This method is quite advantageous when the mechanical system being investigated contains elements which are connected to and remain in physical contact to one another. An example is given and thoroughly discussed to show the effectiveness of the procedure and the display produced.

3. GENERATION OF FIGURES BY INTEGRATION

The figure (F_i, G_i) was required to be continuous, but the waveforms $F_i(t)$ and $G_i(t)$ do not necessarily have continuous derivatives with time. (F_i, G_i) is, in essence, a drawing that can be made with a pencil on paper without removing the pencil from the paper at any time. Lines may be retraced, however, to maintain continuity. This requirement suggests the use of a pair of integrators to generate the waveforms, since a property of the integration of discontinuous waveforms is that the output is continuous. As will be shown, rotating and translating components within a figure are easily accomplished.

To clarify the discussions which follow, consider Figure 3, which shows a 3-component figure that

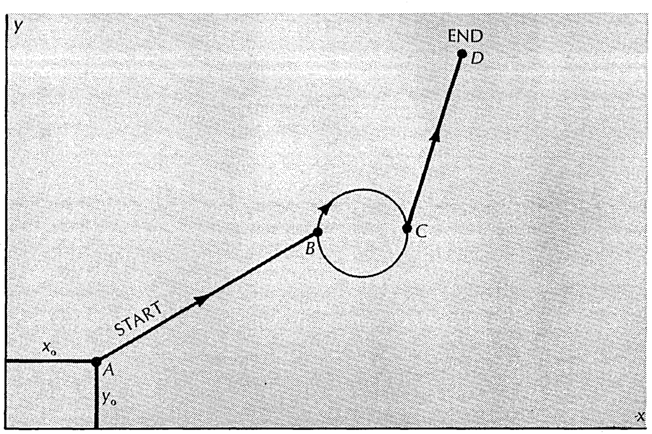


Figure 3 — A 3-Component Figure

may be drawn continuously and represents, perhaps, a mass with connecting rods at both ends. Such a figure can be generated as follows:

1. Use two integrators labelled x and y, with initial conditions x_o and y_o.

2. Start by integrating constants:

$$\dot{x} = C_1$$

$$\dot{y} = C_2$$

The ratio of C_2 to C_1 determines the slope of the line AB. At time t_1 corresponding to point B, switch out C_1 and C_2.

3. At time t_1, switch in

$$\dot{x} = C_3 \sin\omega(t - t_1)$$

$$\dot{y} = C_3 \cos\omega(t - t_1)$$

the solution of which is a circle where C_3 and ω determine the radius. Let the integration proceed until the circle is drawn 1½ times, and switch out both inputs when the values of x and y correspond to point C. Let the switching time be t_2.

4. At time t_2, switch in

$$\dot{x} = C_4$$

$$\dot{y} = C_5$$

to draw line CD, and terminate the integration when point D is reached.

The waveforms of each of the m_i components of the n figures (F_i, G_i) which make up the final picture (x, y) will be referred to as (F_{ij}, G_{ij}) $j = 1, 2, \ldots m_i$, $i = 1, 2, \ldots n$, with the inputs to the integrators as $(\dot{F}_{ij}, \dot{G}_{ij})$. The initial conditions of (F_i, G_i) will be written (F_{io}, G_{io}). The differential equation describing the figure (F_i, G_i) is now given by

$$(\dot{F}_i, \dot{G}_i) = \sum_{j=1}^{m_i} (\dot{F}_{ij}, \dot{G}_{ij}) \quad IC = (F_{io}, G_{io}) \quad (3)$$

where the operation "add" will be taken to mean a "switch" operation, *i.e.*, during the time $t_{j-1} \leq t \leq t_j$, $(\dot{F}_{ij}, \dot{G}_{ij})$ are the inputs to the pair of integrators.

4. THE CONTROL OF INTEGRATION FIGURES

Figures generated by the integration process have some interesting properties. The first to be discussed will be that of rotating a figure, say the i^{th} figure (F_i, G_i), counter-clockwise through an angle θ about the starting point (F_{io}, G_{io}). [See Figure 4.] For convenience the subscript i is dropped in this discussion.

From (1) and (2), the new figure (F', G') must be such that

$$F' = F_o + (F - F_o)\cos\theta - (G - G_o)\sin\theta \quad (4)$$

$$G' = G_o + (F - F_o)\sin\theta + (G - G_o)\cos\theta \quad (5)$$

from which

$$\dot{F}' = \dot{F}\cos\theta - \dot{G}\sin\theta \quad (6)$$

$$IC = (F_o, G_o)$$

$$\dot{G}' = \dot{F}\sin\theta + \dot{G}\cos\theta \quad (7)$$

The rotation transformation may be applied to the inputs of the integrators.

It is clear also that when rotation equations are applied to only some of the inputs generating the components in a figure, only these components are rotated, with successive components translated. Thus, Figure 5 is a new figure derived from Figure 3 when the rotation transformation is applied only to the inputs $[C_3 \sin\omega(t - t_1),\ C_3 \cos\omega(t - t_1)]$ producing the circle. The slope of line CD remains constant, although the line itself is translated. Generating figures by integration is therefore useful whenever components must always be linked to one another at the same place.

For rotations about points other than starting points, only the initial conditions of the integrators need be calculated. This is also the case for simple translations.

To simplify the notation involved in (6) and (7),

$$R_o(\dot{F}, \dot{G})$$

is used to denote the set of inputs to the integrator given by (6) and (7). It is clear that

$$\sum_{j=1}^{m_i} R_o(\dot{F}_{ij}, \dot{G}_{ij}) \equiv R_o[\sum_{j=1}^{m_i} (\dot{F}_{ij}, \dot{G}_{ij})] \quad (8)$$

A linear transformation is useful to enlarge, shrink, stretch, or compress a figure or components within a figure. Suppose a new figure (F', G') is desired with initial conditions (F_o, G_o) such that

$$F' = F_o + a(F - F_o) \quad (9)$$

$$G' = G_o + b(G - G_o) \quad (10)$$

By differentiating

$$\dot{F}' = a\dot{F} \quad (11)$$

$$IC = (F_o, G_o)$$

$$\dot{G}' = b\dot{G} \quad (12)$$

Thus, Figure 6 shows the basic figure (Figure 3) when $1 > a = b > 0$ (Shrinking). Clearly (11) and (12) can be used to distort the basic figure by any amount in either the x or y directions. If $a = 1$ and $b \neq 1$, compressing or stretching occurs. Negative values of a or b (or both) produce mirror images about axes

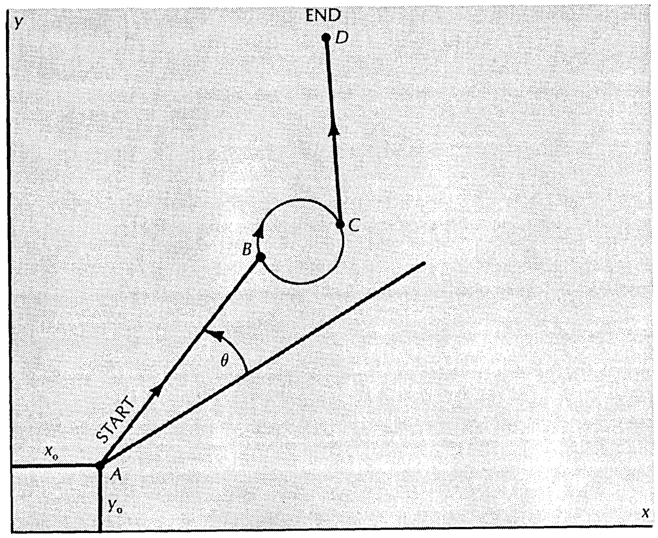


Figure 4 – Rotating the Figure an Angle θ About A

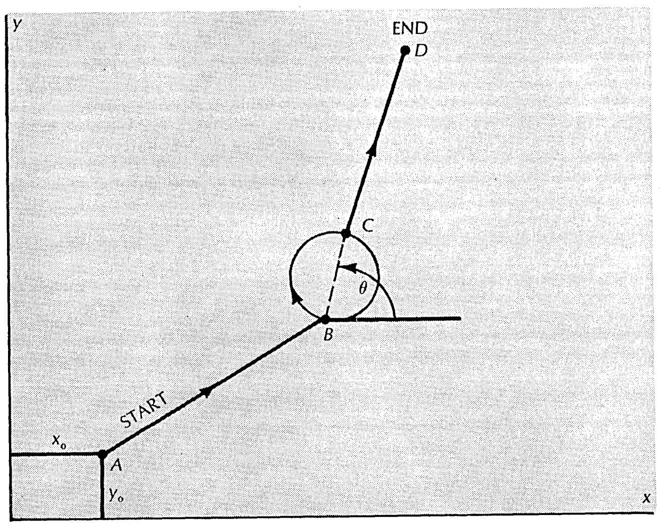


Figure 5 – Rotating Only the Circle

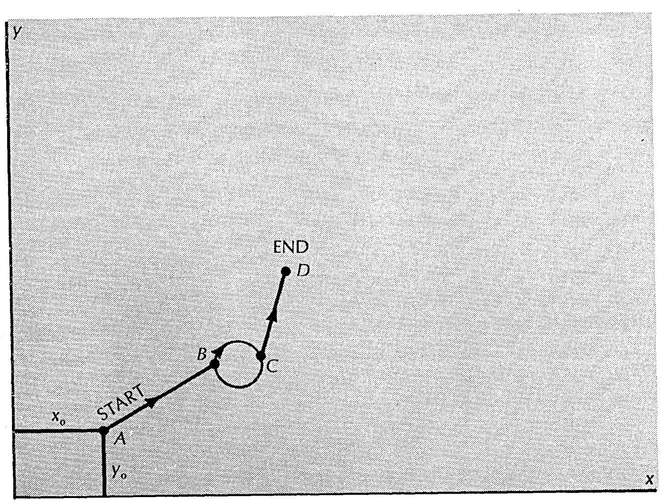


Figure 6 – Shrinking the Figure About Point A

through the starting point and parallel to the original axes. The transformation can also be applied to only some of the components of the figure. (Circles made into ellipses, etc.). To simplify notation of (11) and (12),

$$L_{a,b}(\dot{F}, \dot{G})$$

will be used to denote the set of inputs given by (11) and (12).

Both operations, $R_{\theta}(\dot{F}, \dot{G})$ and $L_{a,b}(\dot{F}, \dot{G})$ are modifications to basic inputs to integrators and are useful in controlling the shape of a final figure. The inputs are explicit functions of time (not of F or G) and it has been assumed that the switching times remain constant. If either operation is applied to the entire input, then the important properties of the figures produced are

1. Parallel lines remain parallel.
2. Closed curves remain closed.
3. The topology of the figure is not changed: Lines remain lines; a four-sided figure is transformed into another four-sided figure, etc.

5. INTEGRATOR INPUTS FOR SOME ELEMENTARY COMPONENTS

With a little imagination, functions of time can be generated to serve as inputs to integrators producing a given figure. Unfortunately, this process sometimes involves a great deal of equipment, but if available can produce excellent displays. The following is a short list of some of the elementary geometrical figures.

A. *A line*

$$(\dot{F}, \dot{G}) = (C_1, C_2)$$

The line can be used as a beam or connecting rod, or simply as an embellishment, such as, a ceiling or table top.

B. *Circle*

$$(\dot{F}, \dot{G}) = (c \sin \omega t, c \cos \omega t)$$

Circles make excellent masses. By stretching or compressing, ellipses may be formed for variation. Cigar-shaped ellipses can serve as long thin masses.

C. *Sine Wave*

$$(\dot{F}, \dot{G}) = (c_1, c_2 \cos \omega t) \text{ or } (c_1 \sin \omega t, c_2)$$

A sine wave is a good representation of a helical spring, since the side view of a stretched helical spring appears as a sine wave.

D. *Parabola*

$$(\dot{F}, \dot{G}) = (c_1, c_2 t) \text{ or } (c_1 t, c_2)$$

This may be used for a flexed beam.

Other conic sections, and even waveforms from non-linear differential equations can be used. However, it is advisable to avoid the latter because of the large amount of equipment that may be required.

E. *Generation of a Square*

An interesting figure which is easily generated by integration is a square, using

$$(\dot{F}, \dot{G}) = (S_x, S_y)$$

where S_x and S_y are shown in Figure 7. Actually the figure produced when S_x and S_y are integrated is a diamond. To form a square with sides parallel to the x- and y-axes, $R_{-\pi/4}(S_x, S_y)$ is formed as in Figure 8:

$$S_x' = .7071 S_x + .7071 S_y$$
$$S_y' = -.7071 S_x + .7071 S_y$$

The new inputs (S_x', S_y') will produce an upright square upon integration. Also, if $L_{a,b}(S_x, S_y)$ is performed before rotation, a rhombus results, which may be changed to a parallelogram by $L_{b,a}(S_x', S_y')$ after rotation.

A square is useful as the side view of a cylinder or a rectangular body.

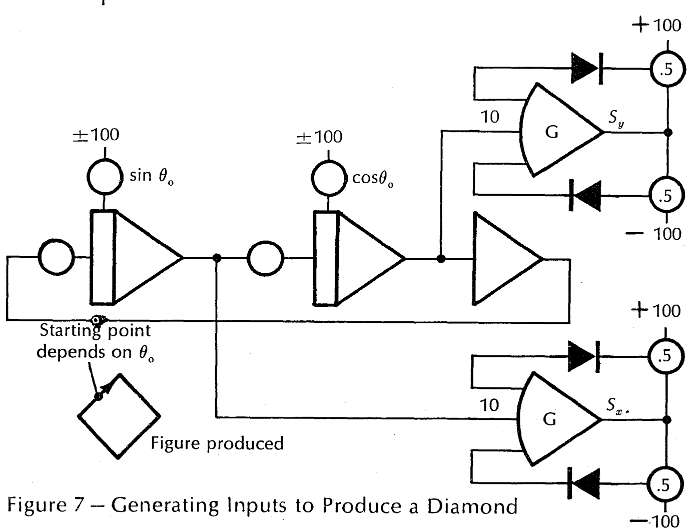


Figure 7 — Generating Inputs to Produce a Diamond

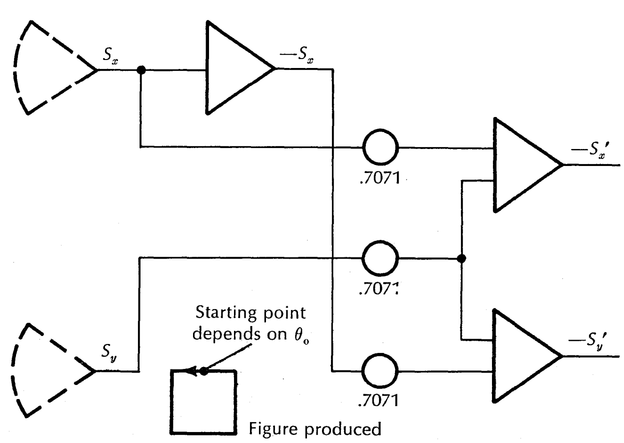


Figure 8 — Rotating the Diamond

6. AN EXAMPLE

To demonstrate the method of generating integration figures, consider a system shown in Figure 9. This system has two degrees of freedom with all motion taking place in a plane. Walls constrain the block to move only vertically, while the pendulum may rotate a full 360° about its bearing attached to the block. A detailed analysis of this system is given elsewhere;[4] it suffices here to say that two coordinates, s and θ, can be obtained from the non-repetitive portion of the computer. The coordinate s is measured from the neutral spring position. When $s=0$, the spring exerts no force. θ will be measured from the vertical in a counter-clockwise direction. Actually, sinθ and cosθ are available from the solution and it is more convenient to use these variables as inputs to the display rather than θ.

A simple one-figure display ($n=1$, $m_1=4$, see Section 3) for this system can be generated as follows:

A. Use a sine wave to represent the spring.

B. An ellipse (or rectangle optionally) can represent the block.

C. A straight line for the pendulum starting at the bottom of the ellipse (or rectangle).

Although the final figure is a good representation of the system, the line for the pendulum lacks the illusion of mass and inertia. Interactions between the block and pendulum would seem unrealistic, so that another component is added.

D. A circle at the end of the line to represent a bob for the pendulum.

Figure 10 shows the proposed drawing with $s=\theta=0$. This may be obtained with the differential equation

$$(\dot{x},\dot{y}) = \left(C_1 \cos \omega_1 t, -C_2\right) + \left(C_3 \cos \omega_2[t-t_1], -C_4 \sin \omega_2 [t-t_1]\right) + \left(0, -C_5\right) + \left(-C_6 \cos \omega_3 [t-t_3], -C_6 \sin \omega_3 [t-t_3]\right) \tag{13}$$

$$IC = (0, y_o)$$

where each C_i, t_i, *and* ω_i is a positive constant. The reader is reminded that the symbol "+" is to be interpreted as a switching operation. It is easier to determine these constants experimentally rather than by calculation.

During the slow time solution of the problem, the spring should appear to change its length in proportion to the value of s. Similarly, the pendulum (line and bob) should be rotated through an angle θ about

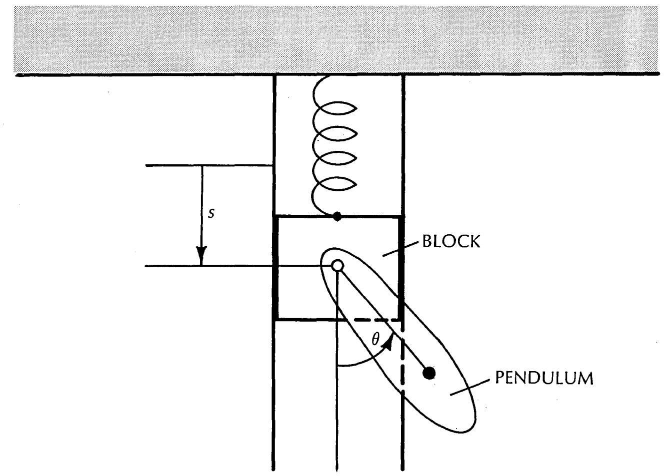


Figure 9 Block & Pendulum

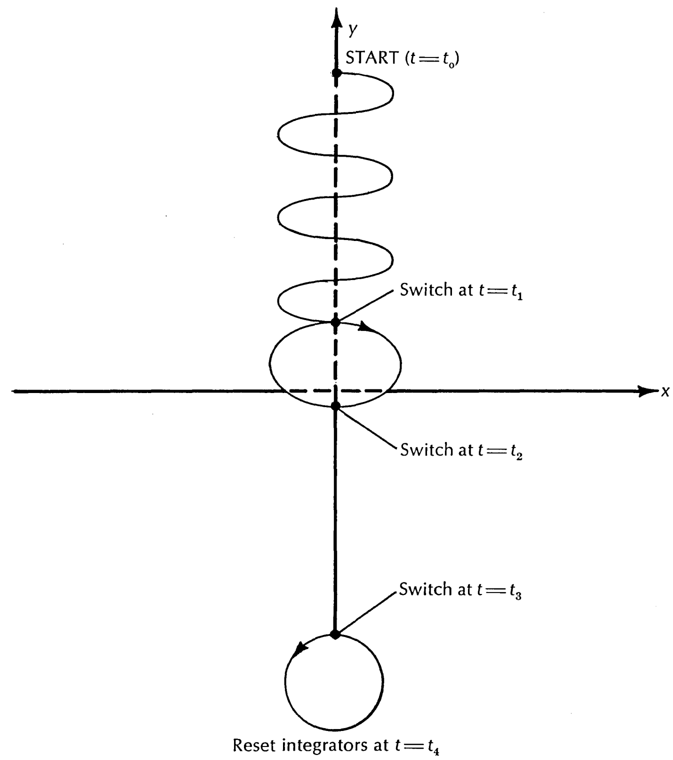


Figure 10 — Drawing of Block and Pendulum System ($s=\theta=0$)

the switch point at $t=t_2$. The complete differential equation is:

$$(\dot{x},\dot{y}) = L_{1,\,1+c_7 s/c_2}\left(C_1 \cos \omega_1 t, -C_2\right) + \left(C_3 \cos \omega_2 [t-t_1], -C_4 \sin \omega_2 [t-t_1]\right) + R_\theta \left[\left(0, -C_5\right) + \left(-C_6 \cos \omega_3 [t-t_3], -C_6 \sin \omega_3 [t-t_3]\right)\right]$$

$$IC = (0, y_o)$$

Using (8) and the definition of $L_{a,b}$ and R_o,

$$
\begin{aligned}
(\dot{x}, \dot{y}) = & \Big(C_1 \cos \omega_1 t, -[C_2 + C_7 s]\Big) \\
& + \Big(C_3 \cos \omega_2 [t - t_1], - C_4 \sin \omega_2 [t - t_1]\Big) \\
& + \Big(C_5 \sin \theta, - C_5 \cos \theta\Big) \\
& + R_o\Big[\Big(- C_6 \cos \omega_3 [t - t_3], \\
& \qquad - C_6 \sin \omega_3 [t - t_3]\Big)\Big] \qquad (15)
\end{aligned}
$$

$$IC = (0, y_o)$$

Generating the last term in (15) can be done with multipliers or a resolver, but rotating a circle is a special case which may be handled differently. Applying (6) and (7) to this term, and using trigonometric laws,

$$
\begin{aligned}
& R_o\Big[\Big(- C_6 \cos \omega_3 [t - t_3], \\
& \qquad - C_6 \sin \omega_3 [t - t_3]\Big)\Big] \\
& = \Big(-C_6 \cos [\omega_3 (t - t_3) + \theta], \\
& \qquad -C_6 \sin [\omega_3 (t - t_3) + \theta]\Big) \qquad (16)
\end{aligned}
$$

With electronic mode control, the comparator controlling the switching at $t = t_3$ can also be used to start an oscillator with initial conditions of $\sin\theta$ and $\cos\theta$. This is, in fact, the method used in the circuit diagram for mechanizing the solution of (15) [Figure 11]. Using 0.01 μf capacitors for the integrators, the required period of 50 msec or less for the repetitive cycle may be obtained with a potentiometer setting for the time ramp of .4 or more. It was necessary to use a high frequency oscillator for the spring in order to obtain several coils. Note that a slight positive feedback was introduced into the oscillator circuit to prevent amplitude decay. The small distortion introduced was unnoticeable in the final figure. This same oscillator is also used to draw the ellipse, since the initial values are correct at $t = t_1$. Figure 11 was drawn with 3 DPDT switches to indicate that switching times control pairs of switches. Actually, 6 SPDT switches like that shown in Figure 12 were used. Such a configuration is easily patched in modern computers (as with EAI's Microstore Units).

A photograph of the final display with $\theta = 90°$ is shown in Figure 13. Patching and adjusting the various constants to obtain the picture desired took several hours but can probably be done very quickly with experience. (The display was set up initially without either mathematical analysis or a circuit diagram.) Figure 14 shows a sequence of pictures produced during a run when s and θ were changing. The net effect is a realistic and striking motion picture of the system.

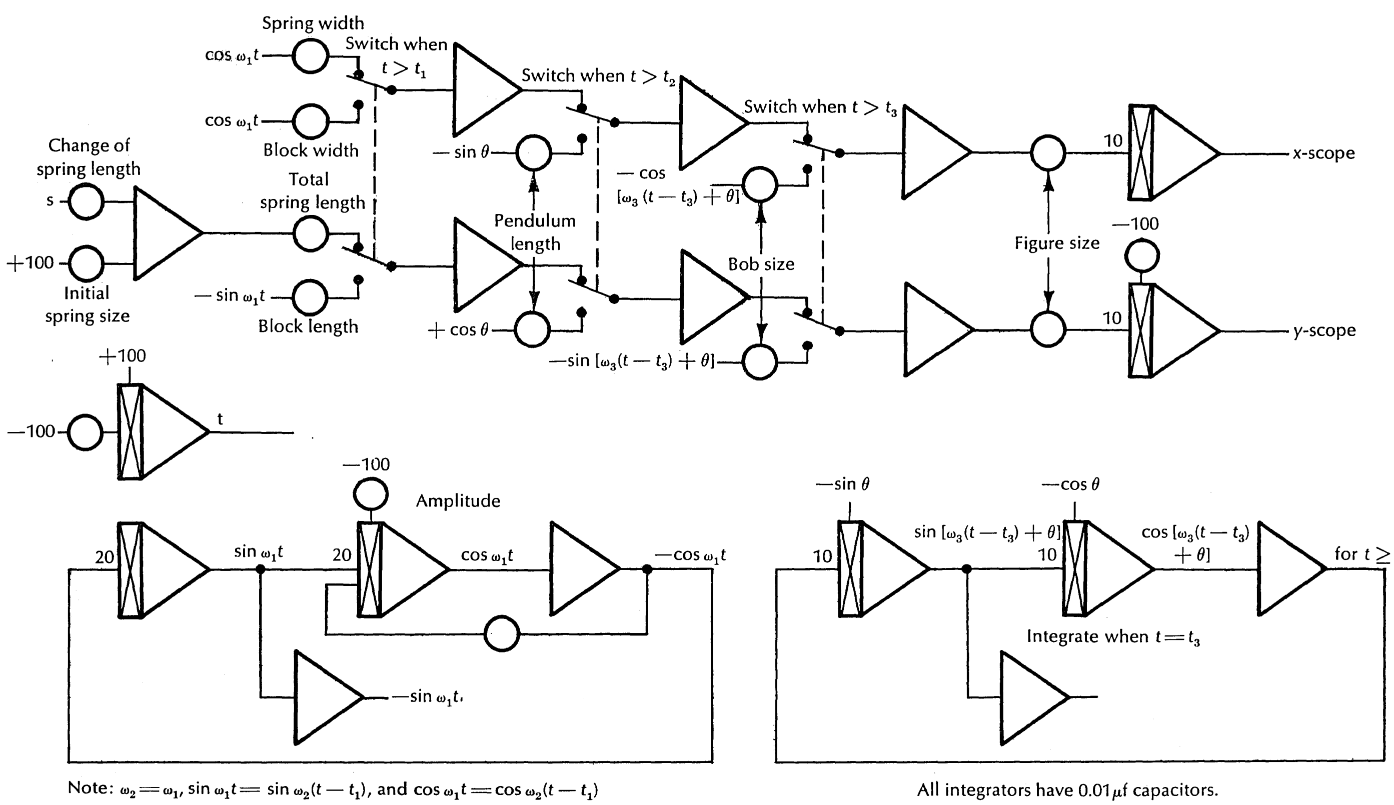


Figure 11 — Circuit Diagram for Block and Pendulum Figure

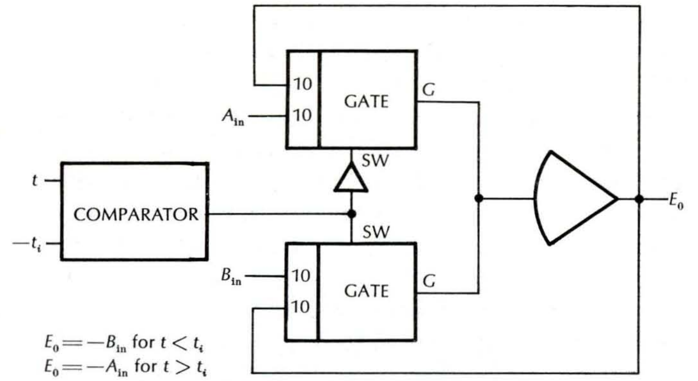


Figure 12 – Electronic SPDT Switch

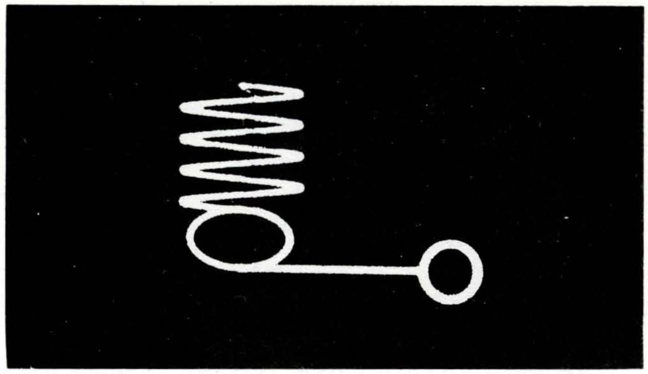


Figure 13 – Scope Picture for $s = -.4$, $\theta = 90°$

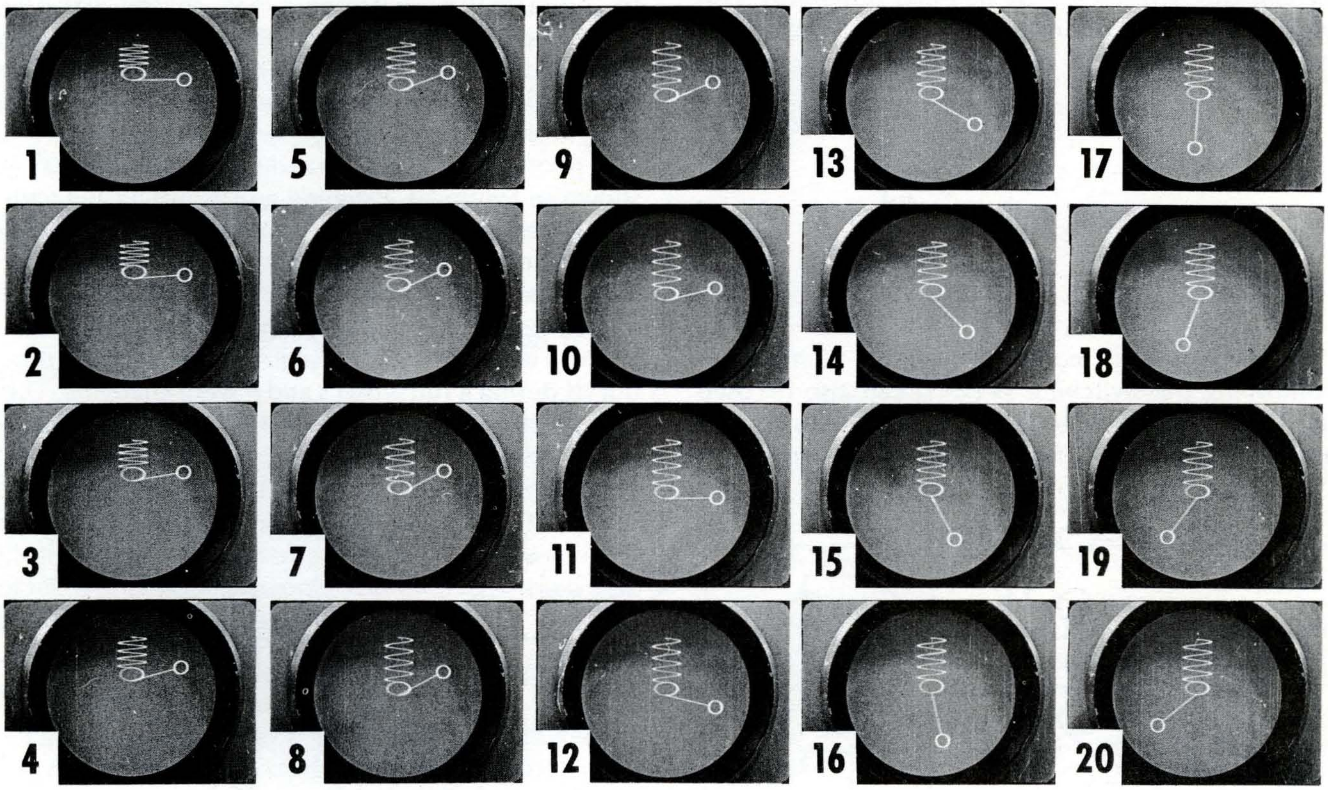


Figure 14 – Sequence of Every 20th Scope Picture

7. CONCLUSIONS

Modern analog computers with electronic mode control and limited digital logic units can be easily programmed and patched to obtain a realistic two-dimensional oscilloscope display showing the motion of a time-varying system. Generating a figure by switching inputs to a pair of integrators is useful if components within the figure remain in physical contact with one another throughout the motion. Translation and rotation of such figures can be programmed easily. Although the process involves much equipment and some effort, the striking results can be beneficial as an educational aid, and provide a great deal of insight into the motion being studied.

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