PROCEEDINGS

Scientific Computation Forum

1948

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H. R. J. GROSCH, *Editor*
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FOREWORD

A Scientific Computation Forum, sponsored by the International Business Machines Corporation, was held in the IBM Department of Education, Endicott, New York, from August 23 to August 26, 1948. The Forum concluded with two sessions held in New York City on August 27.

Earlier meetings in this series, which began in 1940, were devoted largely to statistical procedures. In the 1948 Forum, for the first time, an attempt was made to cover many of the fields in which large-scale computing methods have proved important. The exchange of ideas between workers in fields as diverse as aerodynamics and physical chemistry proved fruitful from the very beginning, yet specialists in the same field also found time for intensive discussions.

It is hoped that the contributions printed here will prove of value not only to the participants but to other members of the growing group engaged in technical calculations on punched card equipment.

CONTENTS

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Evaluation of Higher Order Differences on the Type 602 Calculating Punch

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SOME of the uses of the higher differences of a given function are:

1. Numerical integration using finite differences,
2. Numerical differentiation using finite differences,
3. Subtabulation or interpolation to a fixed interval,
4. Location of errors in a given set of data,
5. Smoothing the irregularities in experimentally obtained data.

This method of evaluating higher differences on the 602 was originally utilized to detect the presence of errors in a computed table of the function x tanh x.

In this circuit, one card containing the value of the function is used for each given value of the argument. As each function card passes through the 602, it has the higher differences as well as the *recomputed* function punched upon it. Since the values of the function and its higher differences at preceding ordinates are required in each computation, the cards must be arranged in their proper sequence. At the beginning of each set of cards, the higher order differences are not directly available. We merely assume that they are equal to zero and therefore the first few values of these differences will be in error.

The recomputed function provides a complete check upon the entire operation of the Type 602 Calculating Punch. The recomputed function will be identical with the given function only if all the crossfooting operations required to solve the set of algebraic equations are correct. This agreement is definite proof that all the calculations are correct.

The comparison circuits of the accounting machine are used to indicate any discrepancy between the given function and the recomputed function. An asterisk is printed adjacent to the error each time a discrepancy occurs. Any machine errors in the differencing process are immediately detected by visually scanning the printed record for the presence of asterisks.

The presence of errors in the given function are also quite evident. An error of magnitude E in the function will affect $(n+1)$ consecutive values of the nth difference by an amount E times the binomial coefficients of $(a-b)^n$. An error of $+1$ affects five fourth differences by amounts of $+1$, -4, $+6$, -4, $+1$. An error of $+1$ affects three second differences by $+1$, -2, $+1$. This characteristic variation of magnitude and sign serves to locate errors immediately.

Mathematical Basis of This Differencing Method

The tabulation of a function and its higher differences shown in Figure 1 illustrates the derivation of the equations used for the computation of the fourth differences.

Function	*Differences* First	Second	Third	Fourth
$f_{-2.0}$				
	$\Delta^{i}_{-1.5}$			
$f_{-1.0}$		$\Delta^{ii}_{1.0}$		
	$\Delta^{i}_{-0.5}$		$\Delta^{iii}_{-0.5}$	
$f_{0.0}$		$\Delta^{ii}_{0.0}$		$\Delta^{iv}_{0.0}$
	$\Delta^{i}_{0.5}$		$\Delta^{iii}_{0.5}$	
$f_{1.0}$		$\Delta^{ii}_{1.0}$		$\Delta^{iv}_{1.0}$
	$\Delta^{i}_{1.5}$		$\Delta^{iii}_{1.5}$	
$f_{2.0}$		$\Delta^{ii}_{2.0}$		
	$\Delta^{i}_{2.5}$			
$f_{3.0}$				

FIGURE 1. TABULATION OF A FUNCTION AND ITS HIGHER-ORDER DIFFERENCES

The first difference may be defined, for instance, by

$$\Delta^{i}_{2.5} = f_{3.0} - f_{2.0} \; ; \qquad (1a)$$

the second difference by

$$\Delta^{ii}_{2.0} = \Delta^{i}_{2.5} - \Delta^{i}_{1.5} \; ; \qquad (1b)$$

the third difference by

$$\Delta^{iii}_{1.5} = \Delta^{ii}_{2.0} - \Delta^{ii}_{1.0} \; ; \qquad (1c)$$

and the fourth difference by

$$\Delta^{iv}_{1.0} = \Delta^{iii}_{1.5} - \Delta^{iii}_{0.5} \; . \qquad (1d)$$

Higher order differences may be expressed by equations of several different forms which are some combination of the formulas given in equations (1). The derivation of an equation for the fourth difference may be given in terms of the function at two given points and the backward diagonal differences from the first point:

$$\begin{aligned}\Delta^{iv}_{1.0} &= \Delta^{iii}_{1.5} - \Delta^{iii}_{0.5}\\ &= \Delta^{ii}_{2.0} - (\Delta^{ii}_{1.0} + \Delta^{iii}_{0.5})\\ &= \Delta^{i}_{2.5} - (\Delta^{i}_{1.5} + \Delta^{ii}_{1.0} + \Delta^{iii}_{0.5})\\ &= f_{3.0} - (f_{2.0} + \Delta^{i}_{1.5} + \Delta^{ii}_{1.0} + \Delta^{iii}_{0.5}) . \qquad (2a)\end{aligned}$$

In addition to the basic equation given for the fourth difference, the corresponding equations for the differences of lower order may be given:

$$\Delta^{iii}_{1.5} = \Delta^{iv}_{1.0} + \Delta^{iii}_{0.5} \qquad (2b)$$

$$\Delta^{ii}_{2.0} = \Delta^{iii}_{1.5} + \Delta^{ii}_{1.0} \qquad (2c)$$

$$\Delta^{i}_{2.5} = \Delta^{ii}_{2.0} + \Delta^{i}_{1.5} \qquad (2d)$$

$$f_{3.0} = \Delta^{i}_{2.5} + f_{2.0} \; . \qquad (2e)$$

These equations are so arranged as to utilize the results of the preceding equation in the computation of the succeeding equation.

Figure 1 and equations (1) and (2) have illustrated the origin and derivation of the formulas which are necessary in the computation of the fourth order differences. However, the principle of this differencing technique may be obtained by a consideration of the simpler equations used for the computation of second order differences which are indicated below. The succeeding description will be devoted to a discussion of these second order difference equations:

$$\Delta^{ii}_{2.0} = f_{3.0} - (f_{2.0} + \Delta^{i}_{1.5}) \qquad (3a)$$

$$\Delta^{i}_{2.5} = \Delta^{ii}_{2.0} + \Delta^{i}_{1.5} \qquad (3b)$$

$$f_{3.0} = \Delta^{i}_{2.5} + f_{2.0} \; . \qquad (3c)$$

Operation of the Differencing Circuit

The time-sequence chart shown in Figure 2 illustrates the flow of information in the Type 602 during a computation of the second order differences. Assuming that the value of the function and its first difference at the preceding argument are available, i.e., if $f_{2.0}$, $\Delta^{i}_{1.5}$ and $-(f_{2.0} + \Delta^{i}_{1.5})$ are in the indicated counters, the second difference is computed as follows:

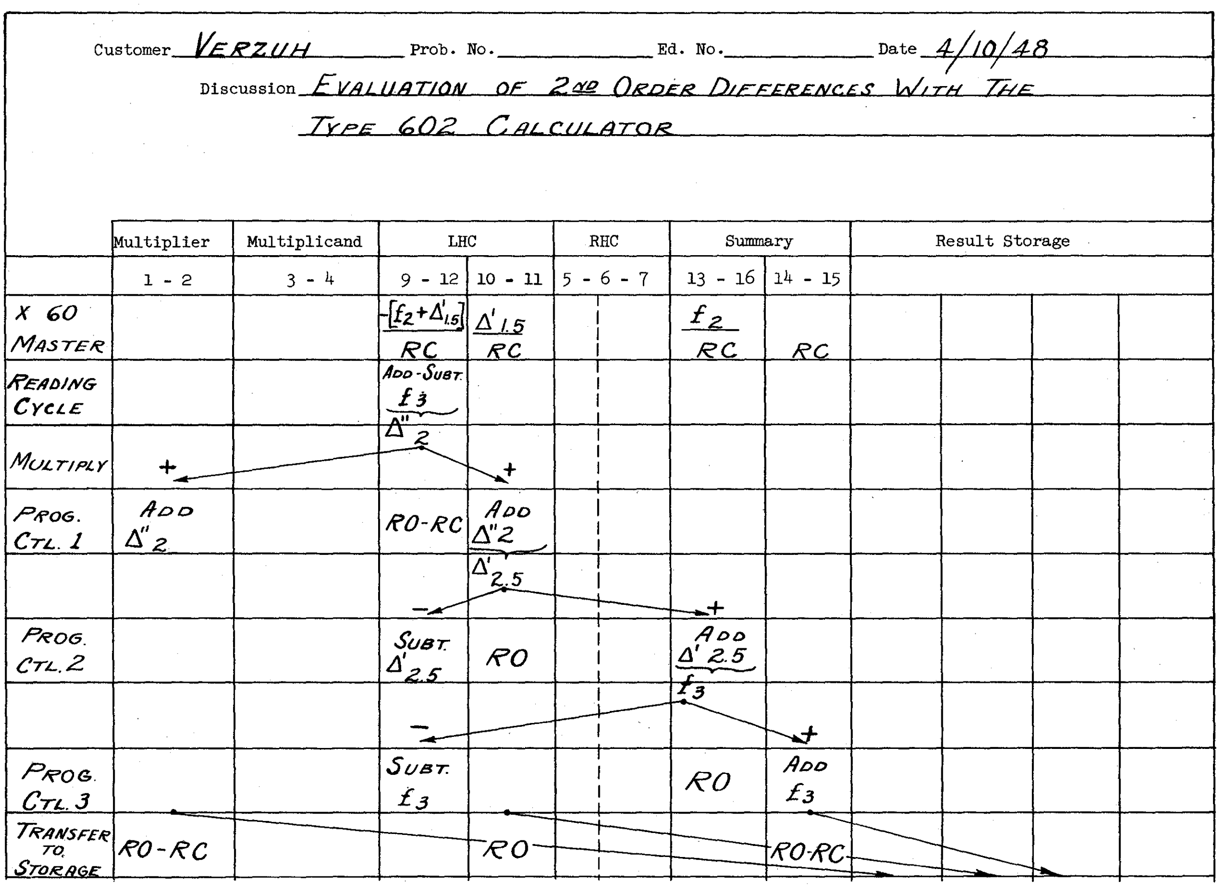

FIGURE 2. FLOW CHART OF 602 COMPUTATION

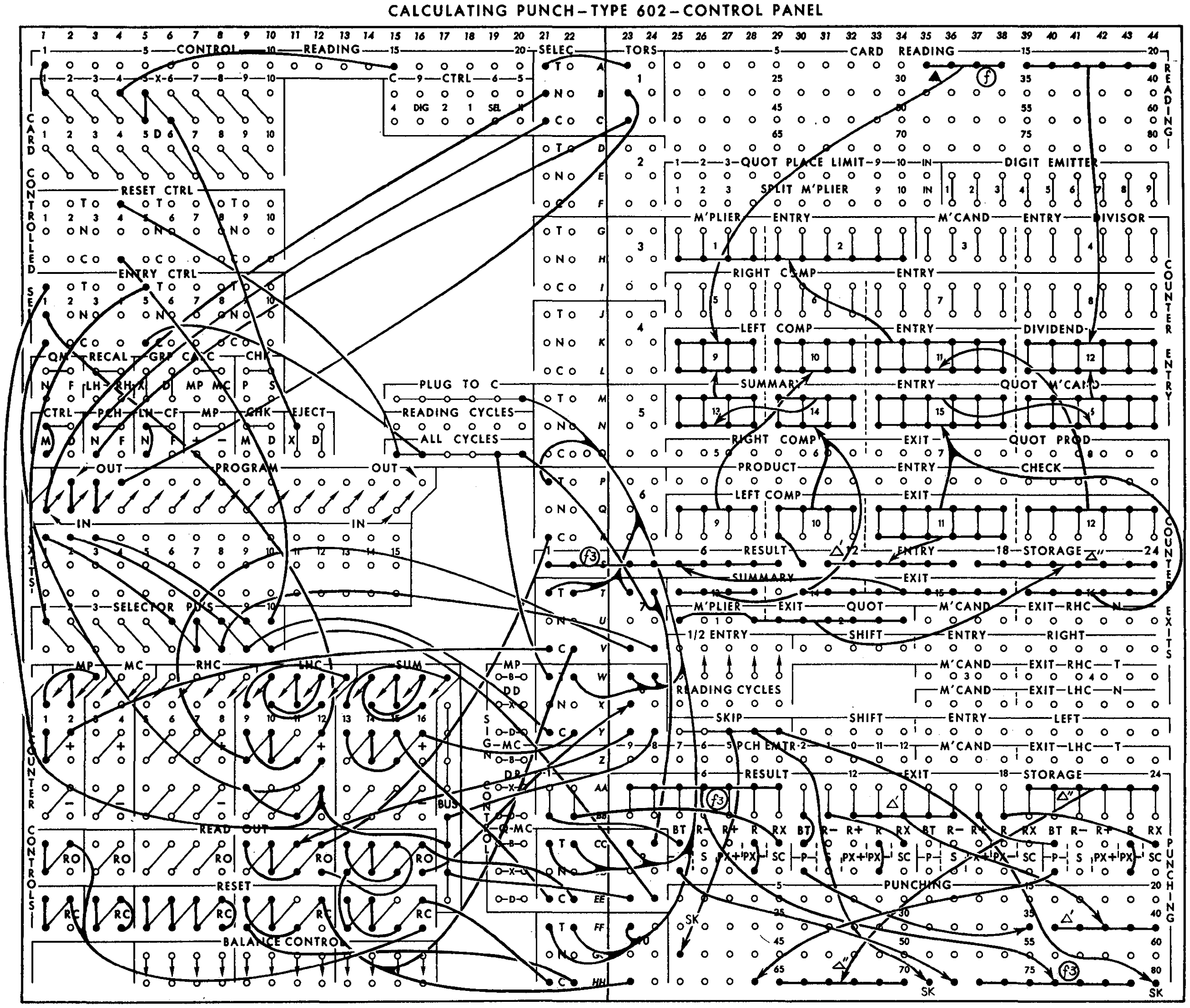

FIGURE 3. WIRING DIAGRAM FOR EVALUATION OF SECOND-ORDER DIFFERENCES

1. The value of $f_{3.0}$ is added to $-(f_{2.0} + \Delta^{i}_{1.5})$ in counters (9, 12) to provide $\Delta^{ii}_{2.0}$ in accordance with equation (3a).
2. This value of $\Delta^{ii}_{2.0}$ is stored in the MP counters (1, 2) and simultaneously added to $\Delta^{i}_{1.5}$ in counters (10, 11) to provide the new first difference $\Delta^{i}_{2.5}$ in accordance with equation (3b).
3. The value of $\Delta^{i}_{2.5}$ is subtracted into counters (9, 12) where it will be available for the computation of the next card. In addition, $\Delta^{i}_{2.5}$ is added to $f_{2.0}$ in counters (13-16) to provide the *recomputed* value of $f_{3.0}$ as indicated in equation (3c).
4. The value of $f_{3.0}$ is subtracted from $-\Delta^{i}_{2.5}$ in counters (9, 12) to provide $-(f_{3.0} + \Delta^{i}_{2.5})$. This quantity is required for the computation of the next card; $f_{3.0}$ is stored in summary counters (14, 15).
5. During the transfer-to-storage cycle the values of $\Delta^{ii}_{2.0}$, $\Delta^{i}_{2.5}$ and the recomputed $f_{3.0}$ are punched on the card containing the original function $f_{3.0}$.

Since this computation is a sequential operation, i.e., the terminal values of the computation of the first card become the initial values for the computation of the next, counters (9, 12), (10, 11), and (13-16) are not reset during the card feed cycle. A master X60 card is used to clear the machine after the differencing of an entire set has been completed. The actual 602 wiring diagram used in the computation of the second order differences is shown in Figure 3.

SERIAL #	f	Δ'	Δ''	Δ'''	Δ^{iv}	f'
1	689 6245	689 6245	689 6245	689 6245	689 6245	689 6245
2	695 6388	6 0143	-683 6102	-373 2347	-062 8592	695 6388
3	701 6543	6 0155	12	683 6114	56 8461	701 6543
→ 4	707 709	6 0256	101	89	-683 6025	707 6799
5	713 6887	6 0088	- 168	-000 0269	-000 0358	713 6887
6	719 7076	6 0189	101	269	538	719 7076
7	725 7278	6 0202	13	-000 0088	-000 0357	725 7278
8	731 7491	6 0213	11	-000 0002	86	731 7491
9	737 7715	6 0224	11		2	737 7715
10	743 7951	6 0236	12	1	1	743 7951
→11	749 8199	6 1260	0 1024	1012	1011	749 9211
12	755 8459	5 9248	- 0 2012	-000 3036	-000 4048	755 8459
13	761 8736	6 0277	0 1029	3041	6077	761 8736
14	767 9013	6 0277		-000 1029	-000 4070	767 9013
15	773 9308	6 0295	18	18	1047	773 9308
16	779 9614	6 0306	11	-000 0007	-000 0025	779 9614
17	785 9932	6 0318	12	1	8	785 9932
18	792 0262	6 0330	12		-000 0001	792 0262
19	798 0603	6 0341	11	-000 0001	-000 0001	798 0603

FIGURE 4. RESULTS OBTAINED DURING EVALUATION OF FOURTH-ORDER DIFFERENCES

The accounting machine is used to provide the printed record shown in Figure 4. This record illustrates the information present on each punched card after it leaves the 602. This record contains a serial number, the given function, the higher order differences in increasing order, and the recomputed function. The two errors present in the functional data are quite evident because of the characteristic appearance of the higher differences.

The flow chart diagram shown in Figure 5 illustrates the flow of information within the 602 during the evaluation of the fourth order differences. The solution of the set of simultaneous equations (2) is effected by the indicated crossfooting operations.

Limitations and Advantages of This Differencing Method

The Type 602 Calculating Punch has been wired to obtain fourth order differences as well as second order differences. Whenever the given function is quite irregular, the higher differences are of appreciable magnitude. In such cases, the available twenty-four positions in the result storage counter are not adequate to punch all of the computed differences. Occasionally, the punching of the first and second differences is omitted, and the available result storage positions are used to punch the large-size higher order differences.

The second order difference board is used for many applications. Whenever the second differences are too large for proper interpretation, the cards are reinserted and the operation is repeated. In such a case, the second order differences are used as an input function, and the fourth order differences are then obtained.

This method of differencing is quite rapid, as differences may be obtained at a rate of 1500 an hour. This number varies slightly with the number of columns punched.

In addition to the error detection properties of this method, it may be easily used for subtabulation, i.e., interpolation to fixed intervals. The 602 may be wired to perform integration using the given value of the function and its backward differences.

Since this method does not require any blank cards for operation, it is capable of a higher operating speed than the methods which do use blank cards.

DISCUSSION

[This paper and the following one by Dr. Gertrude Blanch were discussed as a unit.]

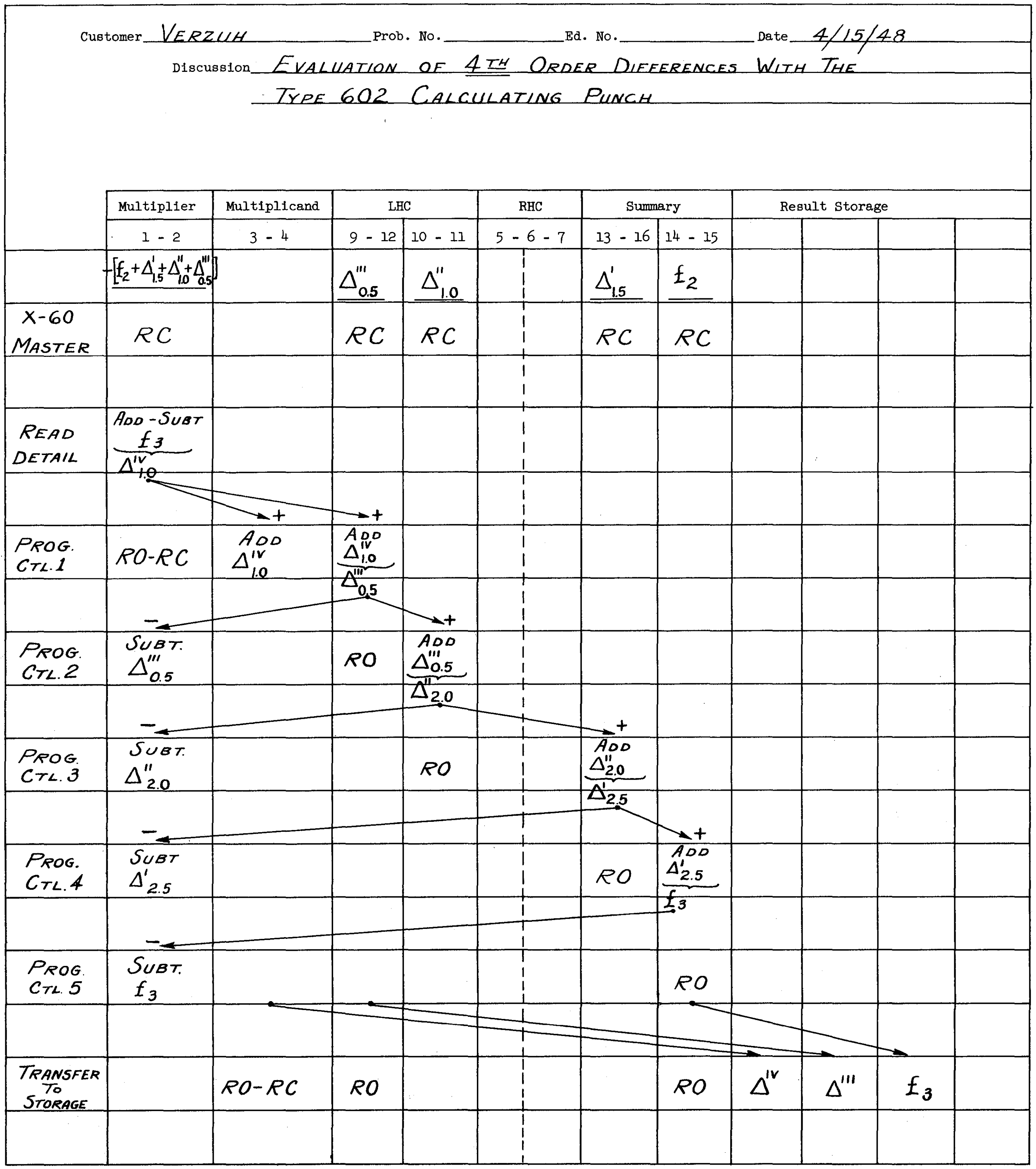

Figure 5. Evaluation of Fourth-Order Differences with the 602

Differencing on the Type 405 Accounting Machine

GERTRUDE BLANCH

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DIFFERENCES have to be taken frequently, not only for the purposes of integrating and differentiating but for the purposes of checking data in the process of computation. If functions are given at uniformly spaced intervals, the process of differencing at strategic stages of the computing process offers a very satisfactory check on the operations.

Now, since differencing has to be done so frequently, it is important to be able to do it on many machines. Sometimes the 602 is tied up on other work and you want to do it on the accounting machine. The 405 was used for differencing long before any other IBM machine. Everybody knows how to take a first difference, summary punch it, and then take the second difference. In this manner successive differences of any order can be built up. But you can also take a sixth difference or fifth difference or fourth difference without taking intermediate differences. This is valuable if you want the differences mainly for checking data. For when you take a high enough difference—the sixth difference for instance, if it is small—you can generally detect the errors by the difference pattern and actually take out the card where the error occurs.

In the summer of 1947, Dr. E. C. Yowell spent his time in our New York laboratory. Dr. Abramowitz had picked up Comrie's paper on getting higher order differences on the National bookkeeping machine, and he asked Dr. Yowell to do something similar on the accounting machine. Dr. Yowell wired a control panel which has been used very successfully, and I will try to give you the wiring of that panel.

(Since the rest of the talk depended heavily on blackboard diagrams and slides, I have taken the liberty to substitute in its place Dr. Yowell's own lucid description of the wiring. It goes into greater detail than my own talk did, and will be much easier to follow for anyone who wants to reproduce the wiring.)

Sixth Differences on the 405

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THE METHOD used in computing the sixth difference is that given by Comrie.[1] The first six functions are used to compute the first through the fifth difference. There is then available in the machine

$$f_6,\quad \Delta^1_{11/2},\quad \Delta^2_5,\quad \Delta^3_{9/2},\quad \Delta^4_4,\quad \Delta^5_{7/2}.$$

The sum of these six quantities is an approximation to f_7, and the difference between f_7 and the sum of these terms will be Δ^6_4. This can be verified by writing

$$f_7 = f_6 + \Delta^1_{13/2}\ ,\qquad \Delta^1_{13/2} = \Delta^1_{11/2} + \Delta^2_6\ .$$

Hence

$$f_7 = f_6 + \Delta^1_{11/2} + \Delta^2_6\ ,\qquad \Delta^2_6 = \Delta^2_5 + \Delta^3_{11/2}\ .$$

Hence

$$f_7 = f_6 + \Delta^1_{11/2} + \Delta^2_5 + \Delta^3_{11/2}\ .$$

This process can be continued until the fifth difference is written as the sum of the previous fifth difference and a sixth difference. Then transposing the equation gives

$$\Delta^6_4 = f_7 - (f_6 + \Delta^1_{11/2} + \Delta^2_5 + \Delta^3_{9/2} + \Delta^4_4 + \Delta^5_{7/2})\ .$$

The machine process is as follows: we assign one counter to the function and one to each difference—seven counters in all. Let us suppose the machine is set up with Δ^6_4 in Counter 1, $\Delta^5_{7/2}$ in Counter 2, Δ^4_4 in Counter 3, $\Delta^3_{9/2}$ in Counter 4, Δ^2_5 in Counter 5, $\Delta^1_{11/2}$ in Counter 6 and f_7 in Counter 7. In the next eight card cycles we will compute Δ^6_5.

During the first card cycle, Δ^6_4 is rolled out of Counter 1 into Counter 2. This addition of Δ^6_4 to $\Delta^5_{7/2}$ gives $\Delta^5_{9/2}$ in Counter 2. At the end of this cycle, a total is taken, printing Δ^6_4 out of Counter 1 and resetting this counter. During the second card cycle, $\Delta^5_{9/2}$ is rolled out of Counter 2 and entered positively into Counter 3 and negatively into Counter 1. This addition of $\Delta^5_{9/2}$ to Δ^4_4 gives Δ^4_5 in Counter 3, while $-\Delta^5_{9/2}$ stands in Counter 1. During the third card cycle, Δ^4_5 is rolled out of Counter 3 and entered positively into Counter 4 and negatively into Counter 1. This addition of Δ^4_5 to $\Delta^3_{9/2}$ gives $\Delta^3_{11/2}$ in Counter 4, while $-(\Delta^5_{9/2} + \Delta^4_5)$ stands in Counter 1. During the fourth card cycle, $\Delta^3_{11/2}$ is rolled out of Counter 4 and entered positively into Counter 5 and negatively into Counter 1. This addition of $\Delta^3_{11/2}$ to Δ^2_5 gives Δ^2_6 in Counter 5, while $-(\Delta^5_{9/2} + \Delta^4_5 + \Delta^3_{11/2})$ stands in Counter 1. During the fifth card cycle, Δ^2_6 is rolled out of Counter 5 and entered positively into Counter 6 and negatively into Counter 1. This addition of Δ^2_6 to $\Delta^1_{11/2}$ gives $\Delta^1_{13/2}$ in Counter 6, while $-(\Delta^5_{9/2} + \Delta^4_5 + \Delta^3_{11/2} + \Delta^2_6)$ stands in Counter 1.

During the sixth card cycle, only the negative transfer into Counter 1 is needed. In the previous cases, we have had to build up our difference order of n from a higher order difference and the previous order of difference n. But the function has been read from the card, so that the function does not have to be built up from the previous function and first difference. Hence, during the sixth card cycle, $\Delta^1_{13/2}$ is rolled out of Counter 6 and entered negatively into Counter 1, thus giving $-(\Delta^5_{9/2} + \Delta^4_5 + \Delta^3_{11/2} + \Delta^2_6 + \Delta^1_{13/2})$ in Counter 1. During the seventh card cycle, f_7 is rolled out of Counter 7 and entered negatively into Counter 1. This gives $-(\Delta^5_{9/2} + \Delta^4_5 + \Delta^3_{11/2} + \Delta^2_6 + \Delta^1_{13/2} + f_7)$ in Counter 1. During this cycle, f_7 is reset as it rolls, leaving the counter open for receiving the next function. This process will be explained later.

During the eighth cycle, f_8 is read positively into Counter 1 and Counter 7. This leaves f_8 in Counter 7 for use in computing the next difference and completes Δ^6_5 in Counter 1, for $\Delta^6_5 = f_8 - (\Delta^5_{9/2} + \Delta^4_5 + \Delta^3_{11/2} + \Delta^2_6 + \Delta^1_{13/2} + f_7)$. During the eight-cycle operation, each counter has advanced from one difference to the succeeding difference of the same order. Hence we are ready to repeat the cycle again and compute Δ^6_6. Since one function card is read every eight cycles, seven blank cards must follow every function card.

The first thing to be done is to set up an eight-cycle control panel. This is done by lacing together seven X distributors (Figure 1).

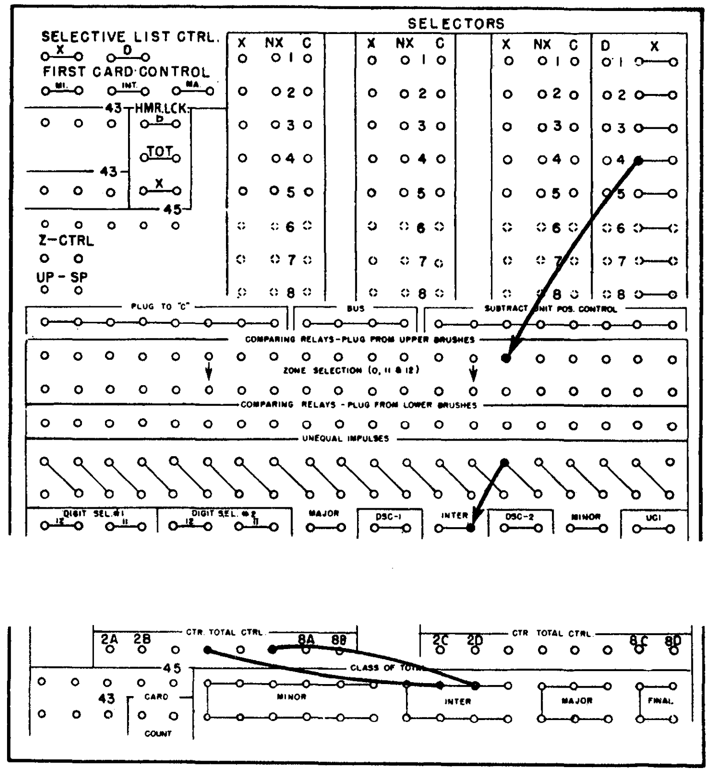

FIGURE 1

As the first card is fed into the machine, the UCI hubs emit impulses at every digit position. The X impulse is selected by the digit selector and passed on to the common hub of Distributor 7. Since 7 is not energized, the impulse comes out of the NX hub and into the common hub of 6. Since this is not energized, the impulse comes out of the NX hub of 6 and enters the common hub of 5. Since 5, 4, 3, 2 are all unenergized, the impulse finally goes from the NX hub of 2 to the common hub of 1. This distributor is unenergized, so the impulse comes out of the NX hub and picks up Distributor 1. Once picked up, the distributor holds for one cycle.

As the second card is fed, the UCI hubs emit impulses for all digits, and the digit selector passes the X impulse along to the distributor chain. It passes along the lacing, as the first impulse did, until it reaches Distributor 1. Since this is still energized, the impulse is shunted to the X hub of the distributor, and from here to the pickup of Distributor 2. Once picked up, this also holds during the next card cycle. As the third card is fed, the UCI hubs emit impulses for all digits, and the digit selector isolates the X impulse and passes it along to the distributor chain. It passes along the lacing until it reaches Distributor 2. Since this is energized, the impulse is directed to the X hub of Distributor 2, from whence it picks up Distributor 3. Notice that the impulse never reaches Distributor 1,

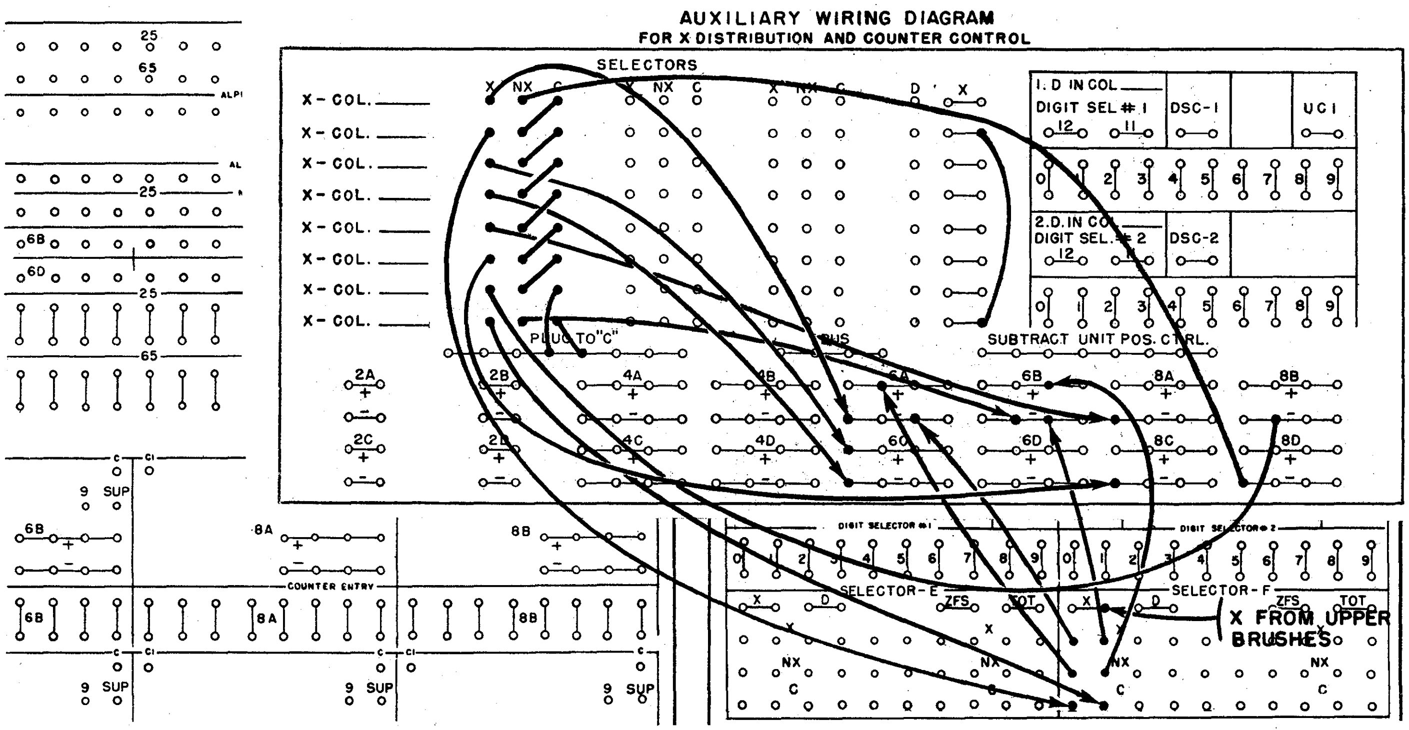

FIGURE 2

since its path is broken by Distributor 2. As the fourth card is fed, the UCI hubs and the digit selector pass an X impulse along to the distributor chain. The impulse passes along the lacing until it reaches Distributor 3. Since this is energized, the impulse is diverted to the X hub of 3, from whence it picks up Distributor 4.

In a similar manner, the fifth card picks up Distributor 5, the sixth card picks up Distributor 6, and the seventh card picks up Distributor 7. As the eighth card is fed, the UCI hubs emit impulses for all digits, and the digit selector passes the X impulse on to the common hub of Selector 7. Since this selector is energized, the impulse is shunted to the X hub of this distributor. As this hub is not wired, the impulse has no effect on the machine, and all selectors are unenergized as the ninth card starts to feed. This is the same condition that we had when the first card started to feed. Thus we have wired a sequence of events which repeats itself every eight cycles.

Seven counters are necessary for a sixth difference computation. This permits handling ten-digit numbers. We shall assign Counter 4A-6A to the function, 4B-6B to the sixth difference; 4C-6C to the fifth difference; 2A-8A to the fourth difference; 2C-8C to the third difference; 2B-8B to the second difference; and 2D-8D to the first difference. All counters are balance coupled, that is, the CI from the highest position is wired back into the C hub of the units position and the "hot 9" is jackplugged to the SUP hub.

An analysis of the desired counter additions and subtractions shows that the following counters must be activated on the indicated cycles.

Cycle	*Roll Positively*	*Onto*	*Roll Negatively Into the Δ^6 Counter*	*Counters*	
1	Δ^6	Δ^5		4B-6B Subt. 4C-6C Add	(1)
2	Δ^5	Δ^4	Δ^5	4C-6C Subt. 2A-8A Add 4B-6B Subt.	(2)
3	Δ^4	Δ^3	Δ^4	2A-8A Subt. 2C-8C Add 4B-6B Subt.	(3)
4	Δ^3	Δ^2	Δ^3	2C-8C Subt. 2B-8B Add 4B-6B Subt.	(4)
5	Δ^2	Δ^1	Δ^2	2B-8B Subt. 2D-8D Add 4B-6B Subt.	(5)
6			Roll Δ^1	2D-8D Subt.	(6)
7			Roll the function	4A-6A Subt. 4B-6B Subt.	(7)
8	Read the new function into the function counter and the Δ^6 counter			4A-6A Add 4B-6B Add	(8)

One cycle of Plug to C impulses is sufficient to control all but the $\triangle^6$ counter. Since any difference counter adds as the next higher difference counter subtracts, only the subtraction needs to be wired to the Plug to C. The addition can be taken care of by wiring 2A-8A add to 4C-6C subtract; 2C-8C add to 2A-8A subtract; 2B-8B add to 2C-8C subtract; 2D-8D add to 2B-8B subtract. The subtraction impulses are generated as shown in Figure 2.

It is at this stage of the wiring that the differencing cycles must be correlated with the card feed cycles. Since we wish to read from the first card, this step must take place as the first card is under the lower brushes. This is the last step in the differencing cycle. As the first card feeds, an X impulse picks up Distributor 1. As the first card passes the upper brushes, the X impulse picks up Distributor 2. Hence this distributor is energized as the first card passes the lower brushes, and the Plug to C impulse causing Counters 6A and 6B to read must come from the X hub of Distributor 2. This fixes the end of the differencing cycle, and the remaining Plug to C impulses are wired in sequence from this point.

Selector F is used for algebraic sign control. Only the reading of the function depends on the punched sign. Once it is entered into the counters as a complement, if negative, or a true figure, if positive, it will always be transferred as a complement or a true figure. Another position of the selector will be used to indicate the sign of the function in the listing process.

The entry of digits into all but the function and sixth difference counters is made by using the card cycle total transfer device. We have already wired the counters to subtract on the cycle when they transfer, and to add on the cycle when they receive. The wiring of the counter exit and counter entry circuits is as follows:

from	4B-6B	Counter total exits.
to	4C-6C	Counter entries.
from	4C-6C	Counter total exits.
to	2A-8A	Counter entries.
from	2A-8A	Counter total exits.
to	2C-8C	Counter entries.
from	2C-8C	Counter total exits.
to	2B-8B	Counter entries.
from	2B-8B	Counter total exits.
to	2D-8D	Counter entries.

The function counter receives impulses only from the brushes:

from		Lower brushes.
to	4A-6A	Counter entries.

The sixth difference counter must receive impulses from all difference counters, the function counter, and the brushes. This demands the use of several selectors. Only five positions will be shown in each selector, but each is to handle the full ten digit field.

Selector A is picked up on the card reading cycle by an X control shot from the pickup hub of Distributor 2 to the X pickup hub of the selector. Selector B is picked up by an X control shot from the pickup hub of Distributor 1 to the X pickup hub of the selector. Selector C is picked up twice; once by an X control shot from the pickup hub of Distributor 4 to the X pickup hub of the selector, and the second time by an X control shot from the pickup hub of Distributor 6 to the D pickup hub of the selector. Selector D is picked up twice; once by an X control shot from the pickup hub of Distributor 5 to the X pickup hub of the selector and once by an X control shot from the pickup hub of Distributor 7 to the D pickup hub of the selector. These impulses are taken from the pickup hubs of the distributors. It must be remembered that the pickup hubs of the selectors and distributors are double hubs so that an impulse wired into one hub is automatically emitted from the other. Hence the selectors and the corresponding distributors will pick up at the same time.

Wiring from the counter list exits is done to avoid the use of split wires from the counter total exit hubs. Whenever a counter is impulsed to add or subtract, the counter list exits and the counter entry hubs are internally connected. Thus, whenever Counter 8A is impulsed to add or subtract, any number reaching the counter entry hubs will also reach the X hubs of Selector C by way of the 8A counter list exits.

The read-in to Counter 6B will proceed as follows: as Distributor 2 picks up, the first card feeds under the lower brushes. Selector A also picks up, so that the entries to 6B are connected by way of the double entry hubs of 6A to the brushes. This reads f into the counter. As Distributor 2 picks up, Distributor 8 picks up also (Figure 2) and the new f is added or subtracted into 6B according to the sign of the function. On the next cycle, Distributor 3 is picked up, but no selector is energized. Hence the 6B entry hubs are connected to the 8B total exit hubs. As only the 6C counter is impulsed on this cycle, no impulses are emitted from 8B into 6B. Also 6B subtracts, so that the difference in 6B is transmitted out through the total exit hubs into the entry hubs of 6C which adds the sixth difference onto the previous fifth difference. On the next card cycle, Distributor 4 is picked up. Hence 6C subtracts and 8A adds, and 6B subtracts through the NX hub of Distributor 8. Selector C is also picked up. Therefore, any information reaching the 8A entry hubs is emitted from the 8A list exit hubs and transferred to the 6B entry hubs. The 8A entry hubs are connected to the 6C total exit hubs. As 6C subtracts, the fifth difference is emitted from the total exit hubs and added into the previous fourth differ-

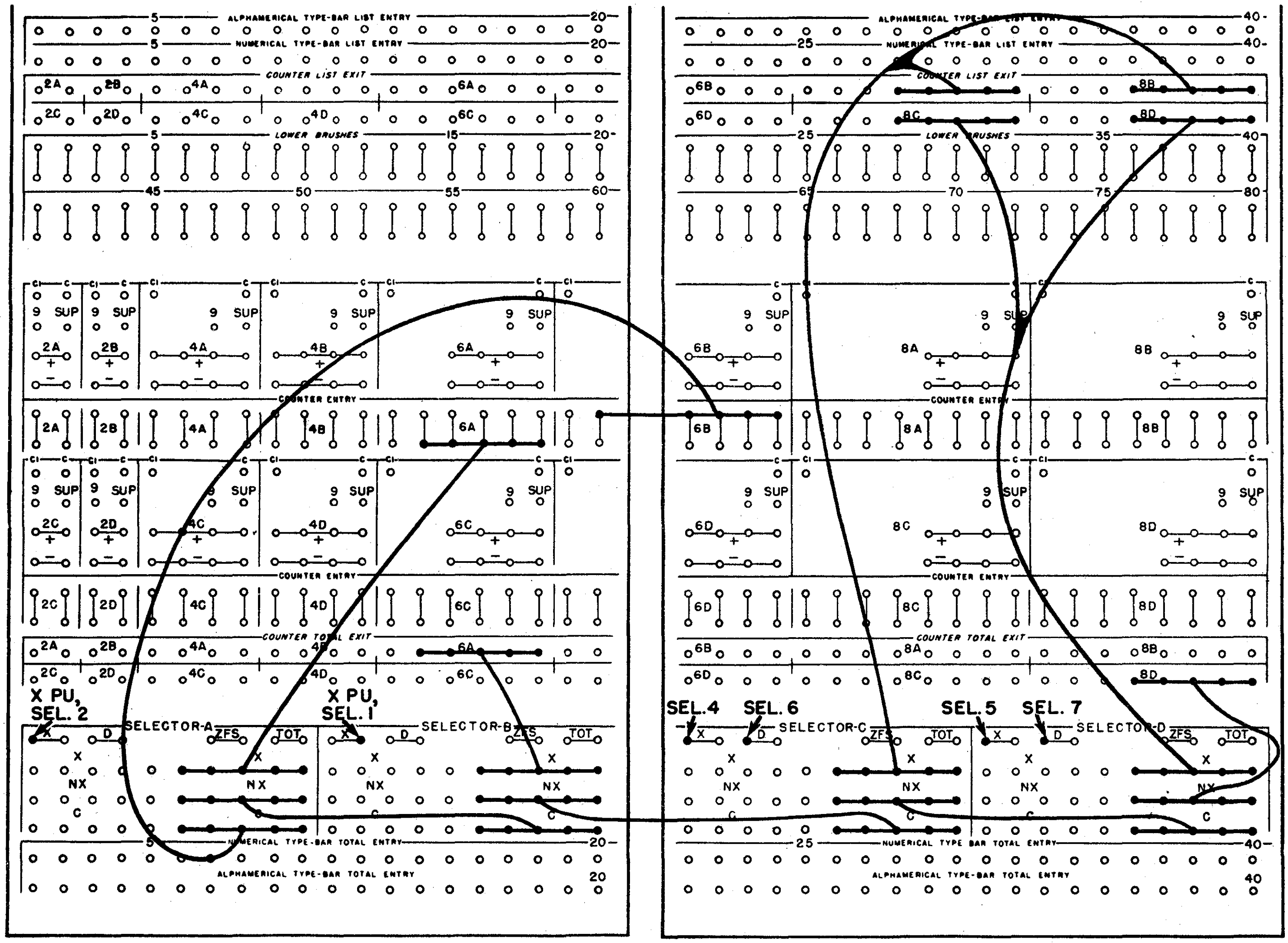

FIGURE 3

ence in 8A. It also passes through 8A and is subtracted in 6B. On the next card cycle, Distributor 5 picks up. Hence 8A subtracts and 8C adds and 6B again subtracts through the NX hub of Distributor 8. Counter 8A transfers the fourth difference into the 8C entry hubs where it is added to the previous third difference.

The fourth difference also is emitted from the list exit hubs and reaches the X hubs of Selector D. Since this distributor is energized, these digits reach the entry hubs of 6B and subtract on top of the fifth difference. On the next cycle, Distributor 6 and Selector C are energized, 8C subtracts, and 8D adds; 6B still subtracts through the NX hub of Distributor 8. Since 8C subtracts, it transmits the third difference to 8B, where it adds to the previous second difference. The third difference is emitted from the 8B list exit hubs. From there, it passes through the X hubs of Selector C into Counter 6B. Note that 8A is not active in this cycle. Hence the 8A list exit hubs are dead, and no confusion can result from double wiring the 8B and 8A list exit hubs to the X hubs of Selector C. On the next card cycle, Distributor 7 and Selector D are energized. Counter 8D adds, while 8B and 6B subtract. Counter 8B emits the second difference, which enters 8D and adds onto the previous first difference. It is also emitted from the 8D list exit hubs and passes through Selector D into 6B. Again the double wiring of the 8C and 8D list exit hubs to the X hubs of Selector D causes no confusion as 8C and 8D are not active at the same time.

On the next card cycle, as no distributor or selector picks up, 8D subtracts and 6B subtracts (Figure 2). Thus 8D emits the first difference from its total exit hubs, and this passes through the NX hubs of Selector D to Counter 6B. On the next cycle, Distributor 1 and Selector A are energized and

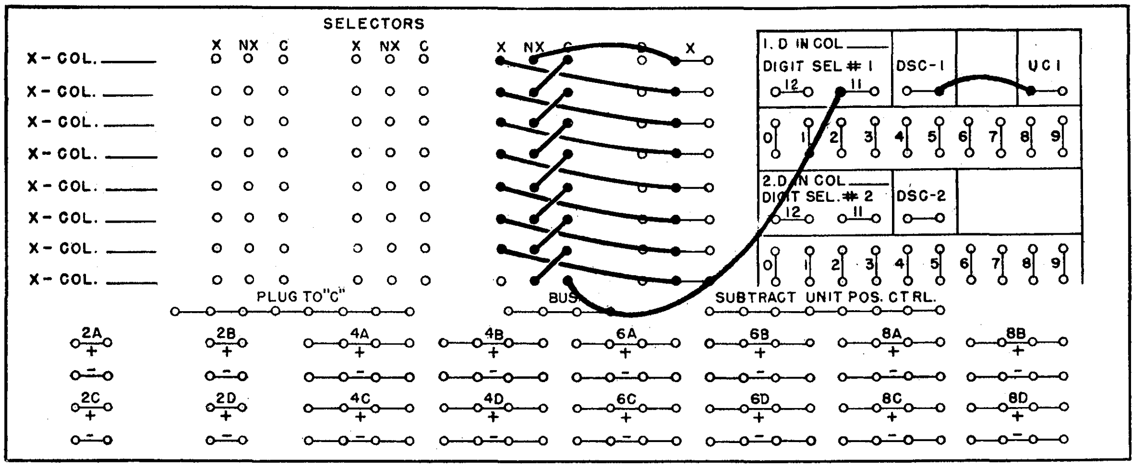

FIGURE 4

6A and 6B subtract. The function is emitted from the total exit hubs of 6A and passes through the X hubs of Selector A into Counter 6B. The next cycle feeds another function card past the brushes and starts the operation all over again.

All counters are to accumulate the numbers read into them except the function and the sixth difference counters. The sixth difference counter should reset after the difference has been rolled out. This is controlled as shown in Figure 4.

The function counter is to be reset after it has transferred its value to 6B. This is best done by rolling the function out of 6A back into 6A (Figure 5).

The pickup hub of Selector E is wired to the pickup hub of Distributor 2. This is to prevent a reset when a negative function is read into 6A. The resetting principle is as follows: 6A is subtracting in order to transfer the information. Hence all its counter wheels are turning. As each wheel passes 9, it emits an impulse which enters the counter through the list exit hub. When an impulse enters a counter position while that counter is subtracting, it stops the wheel. Consequently, the counter wheel is stopped as soon as it emits an impulse—when it stands at 9. A counter with 9 in every position is a counter containing zero, if a "nines complement" system is used. Thus the function counter is reset as it transfers.

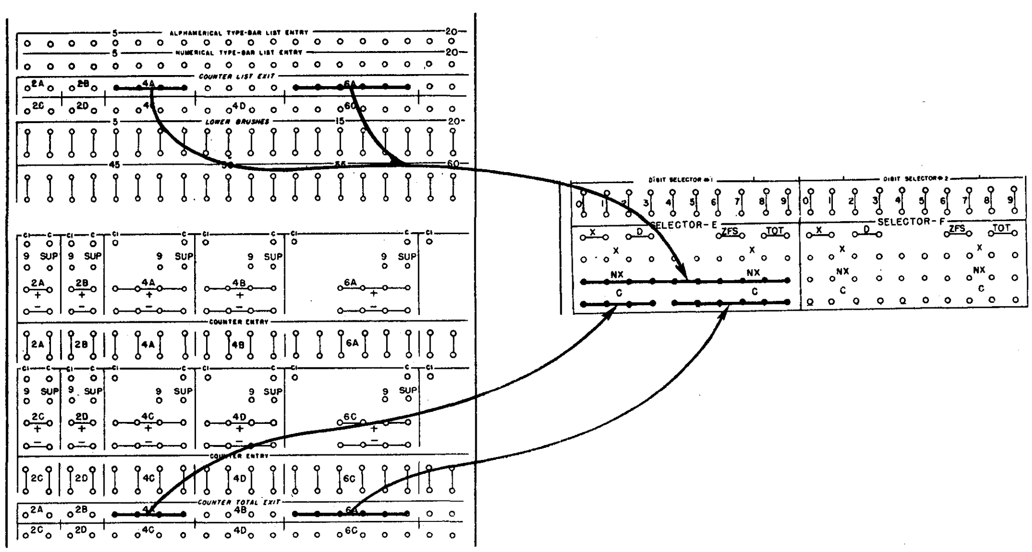

FIGURE 5

To print the sixth difference and list the argument and function, the following wiring is used (Figure 6):

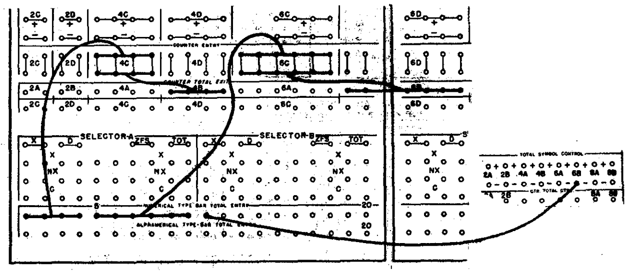

FIGURE 6

Counters 4B-6B are wired for balance conversion (Figure 7):

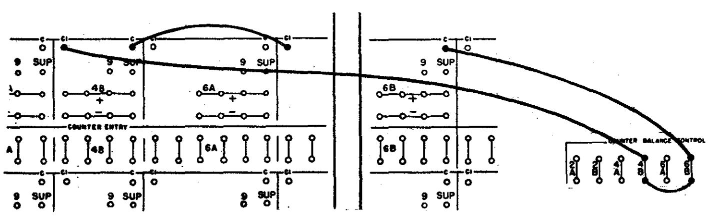

FIGURE 7

The argument and function are listed (Figure 8):

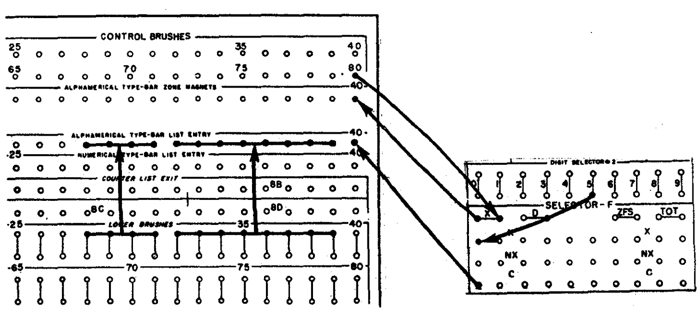

FIGURE 8

This will list an N after each negative function. If a numerical type bar is used for the sign of the function, read one Subtract Units Position into Selector F instead of a "hot 5."

To get the listed numbers printed, a list cycle must be introduced before the function card reads. This is done by wiring the pickup of Distributor 2 to a control position of the comparing relays, and connecting the unequal impulse to the minor control hubs.

When there is one intermediate and one final total each cycle, the paper is spaced three times. In order to bring all the listing and total printing for a single step onto a single line, the upspace suppress is wired from first card control intermediate. This wiring kills spacing on the intermediate total cycle and allows only the single upspace on the minor listing cycle.

In order to compute an nth difference, an $(n + 2)$-cycle panel is needed and $(n + 1)$ blank cards must be inserted between function cards. Higher differences than the sixth can be computed on this same scheme if counter capacity and distributor capacity are available. No extra selector capacity is needed, as Selectors C and D can be multiwired on the same scheme as indicated here.

If the function and argument are read into counters instead of being listed, they can be summary punched together with the highest order difference. An intermediate summary punch control will be sufficient. Since the argument and function can be total printed at the same time as the difference, the listing cycle is not necessary and the minor control break can be eliminated. The intermediate control can then be shifted to minor control, saving one cycle for each difference.

The machine is cleared on the last card by wiring all differencing counters to major total. Since a last card causes all these total breaks, this will reset all counters and clear the machine.

REFERENCE

1. L. J. COMRIE, "On the Construction of Tables by Interpolation," *Monthly Notices Royal Astron. Soc.*, 88 (1928), pp. 447-59 and 518-22.

DISCUSSION

Mr. Ferber: Do you insert blank cards?

Dr. Blanch: Yes, seven blank cards behind every function card, otherwise you couldn't do the operation. The cards are put in by the collator.

Mr. Bisch: Are they absolutely blank cards?

Dr. Blanch: Absolutely blank, and they come out blank. You can use them over and over again. You are really programming the accounting machine.

Dr. Abramowitz: I would like to mention that the sixth difference control panel described by Dr. Blanch may be used for the computation of differences of lower order if so desired. To illustrate this property it is instructive to consider the following operations which take place to produce the successive differences. Let us consider a table of values $f_1, f_2, \ldots$ in which we have interspersed seven blank cards between successive function cards. If the cards have been fed into the machine and the f_1 card is at the lower brushes, we then compute as shown on page 21.

	Δ^6 Counter	Δ^5 Counter	Δ^4 Counter	Δ^3 Counter	Δ^2 Counter	Δ^1 Counter	Function Counter
f_1 card	$+f_1$	0	0	0	0	0	$+f_1$
Blank card 1	Clears	$+f_1$					
Blank card 2	$-f_1$		$+f_1$				
Blank card 3	$-2f_1$			$+f_1$			
Blank card 4	$-3f_1$				$+f_1$		
Blank card 5	$-4f_1$					$+f_1$	
Blank card 6	$-5f_1$						
Blank card 7	$-6f_1$						Clears
f_2 card	f_2-6f_1						$+f_2$
Blank card 1	Clears	f_2-5f_1					
Blank card 2	$-f_2+5f_1$		f_2-4f_1				
Blank card 3	$-2f_2+9f_1$			f_2-3f_1			
Blank card 4	$-3f_2+12f_1$				f_2-2f_1		
Blank card 5	$-4f_2+14f_1$						
Blank card 6	$-5f_2+15f_1$					$f_2-f_1=\Delta^2_1$*	
Blank card 7	$-6f_2+15f_1$						Clears
f_3 card	$f_3-6f_2+15f_1$						$+f_3$
Blank card 1	Clears	$f_3-5f_2+10f_1$					
Blank card 2	$-f_3+5f_2-10f_1$		$f_3-4f_2+6f_1$				
Blank card 3	$-2f_3+9f_2-16f_1$			$f_3-3f_2+3f_1$			
Blank card 4	$-3f_3+12f_2-19f_1$				$f_3-2f_2+f_1=\Delta^2_1$		
Blank card 5	$-4f_3+14f_2-20f_1$					$\Delta^1_1+\Delta^2_1=\Delta^1_2$	
Blank card 6	$-5f_3+15f_2-20f_1$						
Blank card 7	$-6f_3+15f_2-20f_1$						Clears
f_4 card	$f_4-6f_3+15f_2-20f_1$						$+f_4$
Blank card 1	Clears	$f_4-5f_3+10f_2-10f_1$					
Blank card 2	$-f_4+5f_3-10f_2+10f_1$		$f_4-4f_3+6f_2-4f_1$				
Blank card 3	$-2f_4+9f_3-16f_2+14f_1$			$f_4-3f_3+3f_2-f_1=\Delta^3_1$			
Blank card 4	$-3f_4+12f_3-19f_2+15f_1$				$\Delta^2_1+\Delta^3_1=\Delta^2_2$		
Blank card 5	$-4f_4+14f_3-20f_2+15f_1$					$\Delta^1_2+\Delta^2_2=\Delta^1_3$	
Blank card 6	$-5f_4+15f_3-20f_2+15f_1$						
Blank card 7	$-6f_4+15f_3-20f_2+15f_1$						Clears
f_5 card	$f_5-6f_4+15f_3-20f_2+15f_1$						$+f_5$
Blank card 1	Clears	$f_5-5f_4+10f_3-10f_2+5f_1$					
Blank card 2	$-f_5+5f_4-10f_3+10f_2-5f_1$		$f_5-4f_4+6f_3-4f_2+f_1=\Delta^4_1$				
Blank card 3	$-2f_5+9f_4-16f_3+14f_2-6f_1$			$\Delta^3_1+\Delta^4_1=\Delta^3_2$			
Blank card 4	$-3f_5+12f_4-19f_3+15f_2-6f_1$				$\Delta^2_2+\Delta^3_2=\Delta^2_3$		
Blank card 5	$-4f_5+14f_4-20f_3+15f_2-6f_1$					$\Delta^1_3+\Delta^2_3=\Delta^1_4$	
Blank card 6	$-5f_5+15f_4-20f_3+15f_2-6f_1$						
Blank card 7	$-6f_5+15f_4-20f_3+15f_2-6f_1$						Clears
f_6 card	$f_6-6f_5+15f_4-20f_3+15f_2-6f_1$						$+f_6$
Blank card 1	Clears	$f_6-5f_5+10f_4-10f_3+5f_2-f_1=\Delta^5_1$					
Blank card 2	$-f_6+5f_5-10f_4+10f_3-5f_2+f_1$		$\Delta^4_1+\Delta^5_1=\Delta^4_2$				
Blank card 3	$-2f_6+9f_5-16f_4+14f_3-6f_2+f_1$			$\Delta^3_2+\Delta^4_2=\Delta^3_3$			
Blank card 4	$-3f_6+12f_5-19f_4+15f_3-6f_2+f_1$				$\Delta^2_3+\Delta^3_3=\Delta^2_4$		
Blank card 5	$-4f_6+14f_5-20f_4+15f_3-6f_2+f_1$					$\Delta^1_4+\Delta^2_4=\Delta^1_5$	
Blank card 6	$-5f_6+15f_5-20f_4+15f_3-6f_2+f_1$						
Blank card 7	$-6f_6+15f_5-20f_4+15f_3-6f_2+f_1$						Clears
f_7 card	$f_7-6f_6+15f_5-20f_4+15f_3-6f_2+f_1=\Delta^6_1$						$+f_7$

*The subscript convention used here differs from that of Yowell in Dr. Blanch's paper.—*Ed.*

To explain the foregoing techniques, let us confine our attention to the sequence of operations occurring with the f_6 card. When the f_6 card is at the lower brushes, it is added into the $\triangle^6$ counter and function counter. On blank card 1 the $\triangle^6$ counter adds to the $\triangle^5$ counter, the amount in the $\triangle^6$ prints, and the counter is cleared. On blank card 2 the amount in the $\triangle^5$ adds to the $\triangle^4$ counter and subtracts from the $\triangle^6$ counter. A similar process takes place on blank cards 2 to 5. On blank card 6 the $\triangle^1$ counter subtracts from the $\triangle^6$ counter. On blank card 7 the function subtracts from the $\triangle^6$ counter. The function counter is cleared on this card by having the amount stored in the counter subtract from itself. This method of clearance eliminates the necessity of having to stop to take a total.

From this point on the pattern described above continues to produce the successive values of the sixth difference. I have only indicated the changes which take place in the various counters. It is clear that if the add impulse to the $\triangle^1$ counter is eliminated we will transmit a zero balance on blank card 7 and the order of the differences in the remaining counters will be reduced by one. Similarly, if the add impulse to the $\triangle^3$ counter were removed, the process described would generate second differences only. If only fourth differences are desired a minor modification of the above process (using six blank cards) will print all the columns of differences as true figures. In the sixth difference control panel only the quantities in the $\triangle^6$ counter print so that the differences in the other counters are carried as complements. If one wishes to list all columns of differences it is necessary to introduce additional counters from which the amounts may be printed as true figures with appropriate sign indication.

In both cases just described it is possible to difference ten-digit function values taking account of algebraic signs. If only fourth differences are desired, the capacity of the counters may be extended to sixteen digits. It is also possible to take second differences of three functions (using four blank cards), or third differences of two functions simultaneously (using five blank cards).

It is clear that the Type 602 Calculating Punch is a superior machine for the calculation of tabular differences since it is faster and the results are punched on cards. Although the results can be punched on cards when differencing on the accounting machine, this procedure involves summary punching. However, when only a printed record of the differences are wanted for the purpose of checking tables, the accounting machine method is desirable. Comparison of the speed of the accounting machine method with that necessary for differencing on a National, Burroughs or Sundstrand machine shows that it is approximately four times as fast except for time consumed in punching the cards. However, once a card file has been prepared, further computations can be made with it, and this usually compensates for the time spent in key punching. The time necessary for interspersing blank cards is never appreciable, and this operation can usually be done at the same time that the differences are being run.

I would like to mention that at the present time we can compute tables on IBM equipment and type them on the card-controlled typewriter. From our experience, this typewriter made one error in typing 35,000 ten-digit numbers. This is work of a high order of accuracy, but for a table maker there is no compromise with perfection. The resulting manuscript must be subjected to various tests. It would be highly desirable if there were a means of preparing a card file from the typed manuscript. This would obviate the necessity of checking by hand, proofreading or repunching a new set of cards to be compared with the original files.

Mr. Hollander: I would like to suggest something about the notation on the diagrams. There ought to be more detail about where the information is coming from. When a counter is indicated by a column, it requires little more writing to indicate in that column that the counter is being impulsed plus or minus. The flow of information through the machine can then be more easily determined, because one knows something is happening in that column.

Dr. Herget: I would like to point out that the chain of X distributors Dr. Abramowitz showed might better be activated by a punch on one of the function cards, because, if at any time there is a machine failure, the 405 will go on with an out-of-phase cycle of eight. If started by the card, each cycle would be independent.

Mr. Hollander: The blank cards could all carry control punches.

Mr. Bell: Another advantage of that is that the board can be used as a general purpose difference computer with the order of the highest difference determined by the control cards used.

Dr. Eckert: There is one comment I would like to make about Dr. Abramowitz' point that we should have automatic means of reading tables so as not to have to key punch them. If you consider the value of a table and the amount of work you put into computing, it doesn't seem excessive to have such a machine. But to key punch the figures from a paper you publish for posterity is a comparatively cheap method of proofreading. Even though you check the plates perfectly the printing may have been bad, so it is very useful to have at least one copy of the tables looked at by individuals as it comes from the press.

Dr. Abramowitz: The difficulty in the printing doesn't detract from the value of a machine for reading back and checking. You want to be able to read back accounting machine records, too.

Dr. Grosch: I think many people here already know it is possible to take second differences on a two-brush accounting machine by flip-flops, without interleaving blank cards.

The Use of Optimum Interval Mathematical Tables

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THE ART of constructing printed tables of mathematical functions is not by any means static. Indeed, of the half dozen great table makers, two have flourished in our time: the late Jean Peters and L. J. Comrie. The requirements of a good printed table are not often explicitly formulated, but most of the Forum members have worked with "good" and "bad" specimens. Not only must a computer constructing a printed table worry about interpolation methods, tabular intervals, and the detection and elimination of errors, but he must consider very carefully the typography, page format, paper quality, and binding. Discussion of these latter items is not often found in the literature; aside from notes in MTAC reviews and the introductory material in the new Chambers tables,[1] the only discussion I have referred to recently is in the Napier Memorial.[2]

Naturally the aim of the table maker is to facilitate the use of his product. In specifying a figure of merit for usefulness, however, one runs some risk of controversy in assessing a "design" in terms of speed of use, reduction of ocular fatigue, and protection against misreading; the relative weighting of these estimates is even more uncertain. The human element is the vital one in hand computing, and it is not surprising that attempts to predict what computers will like have led to rather varied results!

The situation is far different when we turn to automatic digital computing equipment on the level, say, of the Type 602. For a given system of input, storage, and output the variables of typography, format, and binding disappear; estimates of the usefulness of a particular table can be made with almost the precision of cost accounting; and machine characteristics become more important than human foibles. It is not surprising that machine methods of calculation have led to a very different formulation of the problem of table design, nor is it surprising that the formulation can be much more exact than opinions about printed tables.

A problem involving table lookup should be specified as completely as possible. As always, one needs to know the function and range of arguments required. An exact statement of error requirements is needed: not just "five figures," but a detailed analysis of the permissible error as a function of the argument. Thus a typical specification might be "no error greater than 2.6×10^{-8} for $x \leqq 1$, nor greater than $2.6x \times 10^{-8}$ for $x > 1$; average error of random interpolated values to be zero." The average number of values to be taken from the table at each use should be known, as should the probable number of times the table will be used (on the work to which its construction cost will be charged).

In addition, the speed and operating cost of the machines involved both in using and in constructing the table must be given, and their storage and sequence capacities. Finally, some sort of estimate may be made of the value of the table designer's own time; this is the factor which makes it unwise to spend a month planning how to save a total of three or four hours of 604 time!

The most satisfactory type of table, given requirements permitting its use, is the critical table. No interpolation is required; final answers are obtained by sorting or collating and gang punching. In printed critical tables, one line is required for each possible value of the function (more if the function is not monotonic in the range tabulated). Values of the argument are constructed so as to correspond exactly to values of the function midway between those actually printed. In the optimum interval methods I have developed, a general form of critical table arises from setting the degree of polynomial approximation equal to zero—general in the sense that the maximum tabular error need no longer be exactly one-half in the last place.

Critical tables, however, are indicated only if the number of values of the function required from each use of the table is large compared to the size of the table. Thus a critical table of the sine function from 0° to 80°, with maximum allowable error 1.5×10^{-4}, will consist of 3283 cards. It obviously would be a good choice if more than a thousand values were required each time the table was used, and it obviously would be a poor choice if less than a hundred values were required.

A table requiring linear interpolation usually is considered next. It is far easier to construct than higher order tables, and much smaller than a critical table. But machine characteristics may be an important factor; if the multiplications involved in the interpolatory process are performed on the 604, for example, quadratic or cubic formulas take no longer to evaluate than linear ones, and it is possible to save table bulk subject to limitations imposed by the storage capacity of the 604 and by the extra cost of computing more complicated tables. On the 602, extra multiplications slow down the operational speed, and a different balance must be struck. On the 601, extra card passes are required; for quadratic interpolation Herget's card reversal technique eliminates the extra control panel, but not the second pass.

There is no point in belaboring this subject of economic decisions too far. I will close this section of my paper by giving one very detailed example of the process, and then pass on to more technical matters. Suppose the estimation procedures to be explained later have been applied to a certain problem, and that the following approximate table sizes resulted:

Critical	18,000 cards	(12)
Linear	1,400 cards	(17)
Quadratic	340 cards	(23)
Cubic	110 cards	(29)
Quartic	60 cards	(35)
Quintic	30 cards	(42)

The numbers in parentheses indicate the number of digits punched on each card of the table, including the argument. The arguments on the detail cards are six-digit numbers.

Further suppose that the equipment available includes sorter, collator, reproducer, and 602A and 604 punches. The relative cost of using these machines is taken as 2, 3, 4, 8, and 16 including operator and overhead (these figures will of course differ from installation to installation, and also will be changed for different machine models and extra accessories). Finally, suppose that an average of 250 values is needed for each use of the table.

In the critical case, we shall do best to use both sorter and collator. First we sort the detail cards on six columns, then we collate these on all six columns with the 18,000-card table, selecting out unnecessary table cards. The merged deck, not much under 500 cards, is passed through the reproducer and gang punched. A final collator run removes the detail cards and reassembles the big table. The costs, in arbitrary units, are respectively 0.3, 3.9, 0.4, and 4.0; the total, 8.6. Because of excessive card handling problems we will adopt 9.0 as our reference figure.

In linear interpolation, collating will still pay in spite of the drastic reduction in table size. We sort the 250 detail cards on four columns, collate on four columns with the 1,400-card table, selecting unnecessary table cards, run the merged deck of say 400 cards through a gang punching operation, separate and restore the table as before, and finally make a multiplication of simple $A \times B + C$ form on each detail card. The costs are 0.2, 0.4, 0.4, 0.4, and 1.4 (Type 602A) or 0.8 (Type 604). Choosing the 604, the total cost is 2.2 units, a considerable saving over the previous critical table.

In quadratic interpolation, it is no longer economical to select out unwanted table cards, since it is cheaper to pull apart the table and detail cards on a single pass through the sorter than to reassemble the table deck by collating, while the extra cost of running a few unnecessary table cards through the reproducer is negligible. Hence, the operations are sorting on four columns, collating without selection (340 + 250 cards), gang punching the whole 590-card merged deck, sorting once for separation, and repeated multiplication of the form $(A \times B + C) \times A + D$ on either 602A or 604. The costs are 0.2, 0.2, 0.5, 0.1, and 2.2 (Type 602A) or 0.8 (Type 604). The total is 1.8 if the 604 is used.

In the cubic case the collator is no longer used, and this will be true for the still smaller tables in higher-order interpolation. The costs are 0.3 for sorting, 0.4 for gang punching, 0.1 for separating, and 3.0 (Type 602A) or 0.8 (Type 604) for interpolation. The total cost is therefore 1.6 units.

The quartic case costs 1.4 units, using the Type 604, since the initial sorting cost drops to 0.2 again and the gang punching drops to 0.3. This total cost of 1.4 arbitrary units will not be substantially reduced by going to higher order interpolation, and in fact the Type 604 has to stop here, as its capacity for storing the multiple coefficients of the quintic and higher tables, reading the detail card and punching the answer is exceeded.

No consideration was given to passing the merged deck through the 604, omitting the gang punching operation. Passing a table card through the 604 costs four times as much as passing either a table or a detail card through the reproducer; therefore gang punching should be omitted only if the size of the table is less than one-third the size of the detail deck. This just begins to be the case for the quartic, and the costs for that case figure $0.2 + 1.0 + 0.1 = 1.3$, undoubtedly the very best that can be done.

If the number of times the job is to be done is great enough to warrant constructing the very complicated quartic table, we may claim that this latter case is the most economical. If figures amortizing the cost of constructing the various tables are added to the above, a final choice can be made.

So much for the economics of special tables; now I want to tell you about the methods we use to design and construct optimum interval tables of various orders.

The idea of expanding the interval is not new.[3,4] The exigencies of hand computation prevented adoption in the past, but the advent of punched card equipment in technical computation immediately brought the matter to the fore. The linear univariate case was discussed in some detail,[5] but even at this level it is possible to increase the permitted interval by 40 per cent.[6] The material for the general case is new.

Let us consider the Besselian interpolation formula

$$f = f_i + n \cdot \Delta^{i}_{i+1/2} + \frac{n(n-1)}{2} \cdot \Delta^{ii}_{i+1/2} + \ldots$$

involving odd and mean even differences. The error of neglecting the third difference is

$$\frac{n(n-1/2)(n-1)}{6} \cdot \Delta^{iii}_{i+1/2} \; .$$

This error is zero at $n = 0$ and $n = 1$, and has two extrema of equal size and opposite sign at $n = 1/2 \pm \sqrt{1/12}$. The extreme error is $\sqrt{3}\,\Delta^{iii}/216$, or about $\Delta^{iii}/125$. If this kind of quadratic interpolation is adopted, the rule for interval would be $\Delta^{iii} = 125\,\epsilon$, where ϵ stands for the maximum allowable error due to neglect of third and higher differences.

If we define the error of approximation of the jth line of a table as

$$f(x) - \sum_{i=0}^{p} A_{ij}\, x^i \; , \quad x_j \leqq x < x_{j+1} \; , \qquad (1)$$

where the A's are the coefficients of the approximating polynomial of degree p, we can write this in n-measure as

$$f(n) - \sum_{i=0}^{p} a_{ij}\, n^i \; , \quad 0 \leqq n < 1 \; , \qquad (2)$$

with

$$n = \frac{x - x_j}{x_{j+1} - x_j} \; .$$

This is ordinarily a polynomial of degree $p + 1$; taking out ϵ we write it as

$$\epsilon \cdot E_p(n) \; . \qquad (3)$$

Of course ϵ may vary from line to line of the table. In the example of $p = 2$ Besselian interpolation, $E_p(n)$ was $n/12 - n^2/4 + n^3/6$.

Without further ado, I will simply state that the maximum possible interval is obtained when the error polynomial $E_p(n)$ is chosen from

$$E_0(n) = 1 - 2n$$
$$E_1(n) = 1 - 8n + 8n^2$$
$$E_2(n) = 1 - 18n + 48n^2 - 32n^3$$
$$E_3(n) = 1 - 32n + 160n^2 - 256n^3 + 128n^4$$

or in general

$$E_p(n) = 1 + (p+1) \sum_{i=0}^{p+1} \frac{(-1)^i}{i} \cdot \frac{2^{2i-1}}{(2i-1)!} \cdot \frac{(p+i)!}{(p-i+1)!} \cdot n^i \qquad (4a)$$

$$= (-1)^{p+1} \cos\left[2(p+1)\cos^{-1}\sqrt{n}\right] \; . \qquad (4b)$$

The identification with Chebyshev polynomials is due to Dr. C. C. Bramble.

The error of approximation is therefore $+\,\epsilon$ at $n = 0$, passes through p extrema (alternately $-\,\epsilon$ and $+\,\epsilon$), and is $(-1)^{p+1}\,\epsilon$ at $n = 1$. This is the error distribution which maximizes the interval $\omega = x_{j+1} - x_j$. Note that the tabular values $\sum A_{ij} x_j^i$ are no longer exact except for rounding error. Instead they are wrong by the maximum allowable amount $+\,\epsilon$.

This material is sufficient to handle certain simple cases. For example, consider the sine function near the origin, and quadratic interpolation. We can replace the sine by the first terms of its series, and write,

$$(x - x^3/6) - (A_0 + A_1 x + A_2 x^2) = \epsilon \cdot (1 - 18n + 48n^2 - 32n^3).$$

Then using $x = n\omega$ and equating like powers of n we find

$$-1/6 n^3 \omega^3 = -32 n^3 \epsilon$$

or

$$\omega = (192\epsilon)^{1/3} \; ;$$

the remaining equations can be solved for the A's and the first line of our table is complete. Next we could write the Taylor series expansion about x_1 and repeat the above to get the next interval $x_2 - x_1$ and the next set of A's; this would go along finely until the term $+x^5/120$ began to bother us.

If one tries this process with a more slowly convergent series for $f(x)$, even the first line of the table will give trouble. The square root function is a case in point; I tried very early in my experiments to construct a table of $\sqrt{1-x}$ to be used in going from sines to cosines, but found that $f(x) = 1 - x/2 - x^2/8 - x^3/16 - \ldots$ converged too slowly for even the first line of a ten-decimal $p = 3$ table.

The clue to a more general approach comes from picturing the error polynomial after it is slightly distorted by the presence of terms of degree greater than $p+1$. The shape is different; the extrema are no longer exactly -1 or $+1$; the positions of the extrema have shifted, *but not much!* Therefore, if we insist that the error be $+\,\epsilon$ at $n = 0$, $(-1)^k\epsilon$ at $n = n_k$, and $(-1)^{p+1}\,\epsilon$ at $n = 1$, where the n_k are the p roots of

$$\frac{dE_p(n)}{dn} = 0 \; ,$$

errors very slightly larger than ϵ may occur in the neighborhood of the n_k; this excess error may be taken care of by making the interval very slightly smaller than theory might indicate.

In general, then, after an interval has been adopted, we compute the function for the $p+2$ arguments corresponding to $n = 0$, the n_k's, and $n = 1$. This is done with the usual table-making precaution: two or three extra decimal places are carried. We then write $p+2$ simultaneous equations in the unknowns $a_0, a_1, \ldots, a_p$, and ϵ:

$$\begin{array}{ll}
a_0 \qquad\qquad\qquad\qquad\qquad + & \epsilon = f(0) \\
a_0 + n_1 a_1 + n^2 a_2 + \ldots + n_1^p a_p - & \epsilon = f(n_1) \\
a_0 + n_2 a_1 + n_2^2 a_2 + \ldots + n_2^p a_p + & \epsilon = f(n_2) \\
\ldots \quad \ldots \quad \ldots \qquad \ldots & \ldots \quad \ldots \\
\ldots \quad \ldots \quad \ldots \qquad \ldots & \ldots \quad \ldots \\
a_0 + n_p a_1 + n_p^2 a_2 + \ldots + n_p^p a_p + (-1)^p & \epsilon = f(n_p) \\
a_0 + \quad a_1 + \quad a_2 + \ldots + \quad a_p + (-1)^{p+1} & \epsilon = f(1)
\end{array}$$

These equations are solved for the a's and for ϵ. If ϵ is less than but nearly equal to its desired value, the interval has been chosen correctly.

Due to the nature of the polynomials $E_p(n)$, we are able to write a general rule for the positions of the extrema:

$$n_k = \sin^2 \frac{k\,\pi}{2(p+1)} \quad ; \quad k = 1, 2, 3, \ldots p. \tag{5}$$

Thus there are no extrema for $p = 0$; for $p = 1$ there is a minimum at $n_1 = 1/2$; for $p = 2$ there is a minimum at $n_1 = 1/4$ and a maximum at $n_2 = 3/4$; for $p = 3$ there are minima at $n_1 = 1/2 - \sqrt{1/8}$ and $n_3 = 1/2 + \sqrt{1/8}$, and a maximum at $n_2 = 1/2$.

It is evident that if we write the above system in matrix notation

$$\mathbf{N} \cdot \mathbf{A} = \mathbf{F} \ ,$$

$\mathbf{N}$ is a square matrix of order $p+2$ whose elements depend only on the choice of p. The inverse of $\mathbf{N}$ may therefore be computed once and for all, for the various values of p, and the unknown column matrix $\mathbf{A}$ formed from

$$\mathbf{A} = \mathbf{N}^{-1} \cdot \mathbf{F} \ . \tag{6}$$

The bottom element of $\mathbf{A}$ is ϵ; from the top down, the others are $a_0, a_1, a_2, \ldots, a_p$. Having obtained the a's, the linear transformation to the desired A's is obvious but laborious, since each line of the table requires a different transformation. Values of $\mathbf{N}^{-1}$ for $p = 0$, 1, 2, and 3 are given in Table I.

TABLE I

$$\begin{bmatrix} +1/2 & +1/2 \\ +1/2 & -1/2 \end{bmatrix} \quad \begin{bmatrix} +3/4 & +1/2 & -1/4 \\ -1 & 0 & +1 \\ +1/4 & -1/2 & +1/4 \end{bmatrix}$$

$$\begin{bmatrix} +5/6 & +1/3 & -1/3 & +1/6 \\ -10/3 & +2 & +10/3 & -2 \\ +8/3 & -8/3 & -8/3 & +8/3 \\ +1/6 & -1/3 & +1/3 & -1/6 \end{bmatrix}$$

$$\begin{bmatrix} +7/8 & +1/4 & -1/4 & +1/4 & -1/8 \\ -7 & +\ 4\sqrt{2} & +4 & -\ 4\sqrt{2} & +3 \\ +14 & -12\sqrt{2} & -4 & +12\sqrt{2} & -10 \\ -8 & +\ 8\sqrt{2} & 0 & -\ 8\sqrt{2} & +8 \\ +1/8 & -1/4 & +1/4 & -1/4 & +1/8 \end{bmatrix}$$

A very important feature of the matrix presentation is that the bottom row of $\mathbf{N}^{-1}$, which is the one used in calculating ϵ from $\mathbf{F}$, has the general form

$$\left[+\frac{1}{2(p+1)} \quad -\frac{1}{p+1} \quad +\frac{1}{p+1} \quad \cdots \quad \frac{(-1)^p}{p+1} \quad \frac{(-1)^{p+1}}{2(p+1)} \right] . \tag{7}$$

This furnishes the definitive estimation process required in choosing an interval, and is a powerful tool in polynomial approximation theory.

While these matrix methods serve well in the final stages of table design and in the actual construction, rougher procedures are valuable in the preliminaries. If we consider the term $i = p + 1$ in (4a), put $n = 1$, and equate it to the $(p+1)$th term of the Taylor expansion for $f(x+\omega)$ we get

$$\omega^{p+1} f^{(p+1)}/(p+1)! = (-1)^{p+1} \cdot 2^{2p+1} \cdot \epsilon \ .$$

Absorbing the sign into ϵ, and remembering the relation between differences and derivatives, we can write

$$\Delta^{(p+1)} = \omega^{p+1} f^{(p+1)} = 2^{2p+1} (p+1)!\, \epsilon \, . \tag{8}$$

The more usual cases are as follows:

$$p = 0 \qquad \Delta^{i} = 2\epsilon$$
$$p = 1 \qquad \Delta^{ii} = 16\epsilon$$
$$p = 2 \qquad \Delta^{iii} = 192\epsilon$$
$$p = 3 \qquad \Delta^{iv} = 3072\epsilon \quad .$$

Suppose we wish to make a linear table of $\ln x$, $1.0 \leqq x < 5.0$, with an overall accuracy of 1.8×10^{-6} throughout, and no requirement on the mean error. First we have to consider the form of the table. I have found from experience that in this sort of problem there is a real gain in making the A_0's seven-decimal numbers, but not more. The A_1's will also be seven decimals. The rounding error of the worst A_0 will not exceed 5×10^{-8}, the rounding error of $A_1 x$ (assuming x to be exact) will not exceed $5x \times 10^{-8}$. The final rounding of an interpolated answer f may introduce errors as large as 5×10^{-7} if the answer is given to six decimals only. Therefore ϵ, the maximum allowable error due to the degree of approximation, must be

$$\epsilon = (125 - 5x) \times 10^{-8} .$$

From (8) we have

$$\omega^2 f^{ii} = (20 - 0.8x) \times 10^{-6} \quad ;$$

if it is desirable to squeeze the final table down to the very smallest size, this equation may be used to calculate the intervals. Usually, however, we would take the worst case ($x = 5.0$) and simply use

$$\omega^2 f^{ii} = 16 \times 10^{-6}$$

or

$$\omega^2 = 0.004(f^{ii})^{-1/2}$$

as the interval rule. For the natural logarithm, $f^{ii} = -x^{-2}$ and hence the very simple result, giving ϵ a negative sign, is

$$\omega = 0.004\,x \quad .$$

As a test we take the last interval, with argument 4.980 and interval 0.020. The column matrix $\mathbf{F}$ is

$$\begin{bmatrix} 1.60542989 \\ 1.60743591 \\ 1.60943791 \end{bmatrix}$$

and application of (7) gives $\epsilon = -1.005 \times 10^{-6}$. At $x = 5.0$ the maximum rounding error from all sources will be 0.80×10^{-6}, so the above value of ϵ is exactly as it should be. The actual tabular values turn out to be

$$A_0 = +0.6074339$$
$$A_1 = +0.2004010 \ ,$$

and the errors at $n = 0$, $n_1 = 1/2$, and $n = 1$ are respectively -0.99, $+1.02$, and -0.99 in units of the sixth decimal. These errors are small because there happens to be no error in A_1 (at least to eight decimals), the rounding error of A_0 was only 1.5×10^{-8}, and the final results were not rounded at all. This favorable combination of circumstances will not hold throughout the table.

As for the size of the whole table, detailed examination using values of ω from 0.004 to 0.019 will show that 430 cards are required. For rough estimates, I use

$$\int_{1.0}^{5.0} dx/\omega = 250 \int_{1.0}^{5.0} dx/x = 250 \ln 5.0 = 403 \ .$$

This, of course, assumes a smooth change of interval, and is usually ten or fifteen per cent low.

It is not practical to describe here the extension of these methods to other problems. I have worked out approaches to such variations as cases with error terms of order $p+2$ instead of $p+1$, cases where the mean error must be zero and p is odd (for p even the above methods suffice), and even the bivariate problem. In general, it is fair to say that optimum interval methods in the latter case are warranted only if the table is to be referred to many hundreds of times, for the work of design and construction is enormous even for the linear case.

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DISCUSSION

[Discussion of this paper was omitted because of time limitations.]

Punched Card Techniques for the Solution of Simultaneous Equations and Other Matrix Operations

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ELECTRICAL punched card equipment has been used for matrix calculations of various sorts for some time. There have been wide discrepancies in the operational methods, efficiency and general utility of the procedures being used. From the widespread interest in this subject, it is evident that there is a genuine need for a good basic approach to the problem, particularly in terms of actual machine operations.

The method explained below has been successfully used for a wide variety of problems over an extended period of time. Among the advantages of the system are the following:

1. Both card handling and machine operations have been reduced to a minimum. This results in a definite time advantage, as well as simplicity from the operator's standpoint.
2. Among the basic matrix operations to which the same procedure is directly applicable, are the following:
   a. Solution of systems of simultaneous equations.
   b. Computation of determinants.
   c. Calculation of inverse matrices.
3. The same method, procedure, and control panels can be used for either real or complex numbers.[1]

The equipment described is that made by the International Business Machines Corporation.

Mathematical Basis of Method

There is a systematic method of operating on a matrix in such a way as to make all of the elements of a given column, except one, equal to zero. The method consists of subtracting some multiple of a given row from each of the other rows in such a way as to make one column zero in all except one element. The process has been well described in the literature[2, 3] and is often called the "starring process." A simple numerical example follows.

Given a set of simultaneous linear equations

$$\begin{aligned} 3X + 12Y - 7Z &= 6 \\ 2X - 8Y + 8Z &= 10 \\ 6X - 2Y - 3Z &= -7 \end{aligned} ,$$

write the matrix

$$\begin{matrix} 3 & 12 & -7 & 6 \\ 2 & -8 & 8 & 10 \\ 6 & -2 & -3^* & -7 \end{matrix} .$$

The first reduced matrix, by "starring" on the A_{33} term, becomes

$$\begin{matrix} 3-\frac{(-7)(6)}{(-3)} & -12-\frac{(-7)(-2)}{(-3)} & -7-\frac{(-7)(-3)}{(-3)} & 6-\frac{(-7)(-7)}{(-3)} \\ 2-\frac{(8)(6)}{(-3)} & -8-\frac{(8)(-2)}{(-3)} & 8-\frac{(8)(-3)}{(-3)} & 10-\frac{(8)(-7)}{(-3)} \end{matrix} ,$$

which reduces to

$$\begin{matrix} -11 & 50/3 & 0 & 67/3 \\ 18 & -40/3 & 0 & -26/3 \\ 6 & -2 & -3 & -7 \end{matrix} .$$

Repetitions of this pattern, working each time with the transformed matrix from the previous operation and starring on one of the remaining main diagonal terms, will produce a diagonal matrix from which the unknowns or the determinant can be computed directly. The basic operation is thus of the form

$$A_{ij} - \frac{A_{in}\,A_{mj}}{A_{mm}} = {}'A_{ij}$$

where $'A_{ij}$ is the A_{ij} term in the transformed matrix and A_{mm} is the starring term in a matrix of order n.

IBM Method

The major problem in matrix work is that a given card, representing a certain element of the matrix, must be handled in a different manner at different steps in the procedure. As an example, what is an answer from a previous transformation might become a multiplier in the next reduction. An obvious solution is to use a master deck which knows the factors to be used, the operations to be performed, and the identity of the resulting term. The actual numerical values for a particular problem are then transferred to the master cards by a relating and gang punching process. A method of this type has been published.[4] The disadvantages of such a process are the necessity of constructing the master files, and the number of unnecessary operations involved.

The key to the procedure explained here lies in a very simple way of causing a machine to differentiate automatically between two identical cards. The usual method of identifying a type of IBM card is with an X punch in all cards of that type. This is a high punch placed in some column of the card and falling at the upper edge. It can be used for controlling the various functions of the different machines through which the card passes. If a card goes through a machine face up rather than face down, all punching in the card is transferred to a mirror image position with right and left transposed. It becomes, to all intents and purposes, a different kind of card. The fact that this requires the machine to read each amount from different columns and from right to left instead of left to right imposes no restrictions. The card form used is shown in Figure 1. The card is in its normal reading position; when the card is in its reversed or mirror image position, the machines sense X58.

Control Panels

The control panels required are as follows:

1. A gang punching control panel which transfers the factors of the starring row into all related cards. The starring row becomes gang punch master cards which are used in the "turned" position. This function is performed automatically by a Reproducing Punch, Type 513 or Type 519, at a constant speed of one hundred cards per minute.
2. A Type 602 Calculating Punch control panel which performs the operation

$$'A_{ij} = A_{ij} - \frac{A_{im}\ A_{mj}}{A_{mm}} \quad .$$

The factors A_{im} and A_{mm} are in "turned" cards which are recognized by the machine by a common X58. The ratio of these terms is computed by a division operation and the result stored within counters in the machine for use on all terms in a particular row of the matrix. This eliminates any redundancy in the division operation, which is the slowest operation performed, requiring approximately four seconds per card.

On an X23 card (normal position) the machine multiplies the ratio by A_{mj} and subtracts it from A_{ij} to form the new term $'A_{ij}$, which is punched into each detail card. The order number of the new matrix term is simultaneously calculated and re-

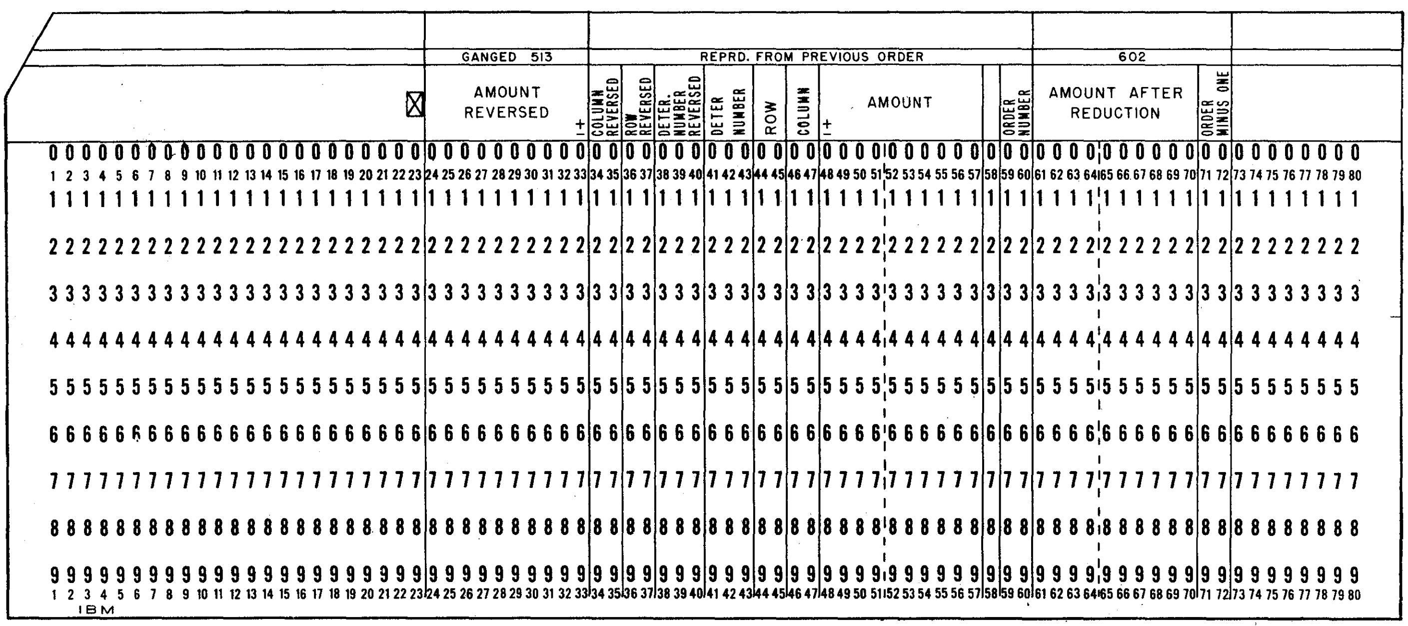

FIGURE 1

corded. The speed here is about twenty-five cards per minute. All of these mathematical manipulations are completely automatic, with the 602 sensing the type of card, determining what is to be read, and performing the proper operations.

3. A reproducing control panel which at a hundred cards per minute will make a new deck with the newly computed transformed elements returned to the original locations in the new cards in readiness for the next starring operation.

Procedure

The detailed steps followed by the operators are exceedingly simple. A complete operating procedure is given below. This covers the calculation of the diagonal matrix. This method, as explained above, will give the solution of simultaneous equations, inverse matrices and determinants.

1. Sort the cards to column 44-45.
2. When column 44-45 equals 59-60, reverse the selected cards, place in front and sort:

   46-47 minor
   41-43 major
3. Gang punch using control panel (1).
4. Select the X58 (turned) cards and hold aside.
5. Sort remaining cards to 46-47.
6. When 46-47 equals 59-60, reverse the selected cards, place in front and sort:

   44-45 minor
   41-43 major
7. Calculate $'A_{ij}$, the elements of the transformed matrix, using board (2).
8. Select X58 cards and hold aside.
9. Reproduce the balance of the cards on control panel (3).

This completes one starring operation. Using the new cards from (9), begin again at step (1) and repeat. Usually it will be adequate to select the "starring" terms by going up the main diagonal.

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DISCUSSION

Mr. Harman: When you use the cards face up, don't you have trouble with curvature?

Mr. Bell: No trouble at all.

Dr. Caldwell: What kind of climate do you have?

Mr. Bell: Los Angeles.

Dr. Herget: We could do it at Washington.

Dr. Grosch: We did it in New York.

Mr. Bell: This approach of reversing the cards will work in many problems. When you reverse the cards in the matrix work, the identification field is reversed also, so it is necessary to have another field where the identification is punched as a mirror image. Then when you turn them over and sort you do not have to invert the sorting procedure or change the sorter brush setting.

Mr. Ferber: Are your elements real in this case?

Mr. Bell: Complex.

Mr. Ferber: In finding characteristic roots, you make a guess from previous experience?

Mr. Bell: Right.

Mrs. Rhodes: You have been very lucky with fiftieth order linear equations. What did you do about loss of accuracy, go on faith?

Mr. Bell: Yes, I went on faith.

Mrs. Rhodes: Did you position for division? If so, how?

Mr. Bell: You are talking about a very real problem. Occasionally we get a matrix that doesn't want to come out. I have yet to find a way by which you can easily and quickly predict that you are about to get into that situation. And we have been working with relatively small problems.

In working large problems the gang punch master cards can be deleted at each stage. Then a straightforward back solution is used after the first unknown is obtained. This does not work well if you have just a few equations. It is more work to do the special back solution than to carry the extra terms along as you go.

Dr. Grosch: There are still only a few installations which have had experience with problems of very large order. We did a problem with forty-one unknowns, carrying the full eight figures permitted by the 601. Our procedure for size was simply that we never starred a number which was less than one-tenth as large as the largest element in its row. We looked at a printed record produced by the 405 while applying the usual check sums. In mass production one could suppress listing for all but the starred row to save paper.

Following that rule, we had to rearrange only once in forty reductions. We lost only two significant figures. How many figures did you lose in your fiftieth order problem?

Mr. Bell: We had four or five decimal accuracy.

Dr. Caldwell: That is about our experience in large cases.

Mr. Ferber: If you want to solve a small set of equations with seven or eight unknowns, the machines aren't adapted to working with such a small number of cards and one comes to the point where it is quicker to do them some other way. Then again, since the work goes up very rapidly with increasing order, you soon come up against a blank wall in that the work is prohibitive. How do you handle the problem?

Mr. Bell: We do enough matrix work to keep the basic 602 panels wired up; so we need not allow time for that. Our operators are familiar with the job, and they handle it efficiently even with only seven or eight unknowns. Everything is kept ready for them. To give you some idea of time, it requires between one and two hours to evaluate an eighth order complex determinant. If there are enough problems, you can split them up so that some are multiplying while others are being sorted and gang punched, with no idle machines.

We have tried to handle special cases by special procedures, but have lost on it. I feel it is better not to use special procedures. Instead we stick to methods that all of our people understand; then we are more certain of coming out with a good answer.

Mr. Harman: I was wondering if you could get around the division step in your formula by the expansion of a second-order minor, which involves the difference between two multiplications.

Mr. Bell: We have tried doing that, and here is the problem we got into: the numbers change tremendously in size, and it is necessary to stop for inspection. That takes longer than the regular method.

Dr. Tukey: The thing is to get away from division by small numbers.

Mr. Bell: Yes. Most of our work is brought in to us. Sometimes we know what is behind the problems, and sometimes we do not. Most of the work is in engineering fields where extreme accuracy is not required. But we do have to calculate with expanded accuracy occasionally.

Dr. Tukey: As I understand the situation, the pessimists thought—and I was one of them—that you lost, roughly speaking, a constant number of significant figures for each new equation. Now it has been proved[1] that it doesn't go like that; the loss goes like the logarithm of the order. If you are thinking of big equations that is a tremendous improvement. I think that size ought not to be taken as grounds for pessimism.

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Two Numerical Methods of Integration Using Predetermined Factors

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VARIOUS systems have been proposed for numerically integrating expressions for which no formal method has been found. Although many of these systems are highly accurate, the adaptability of some of them to mechanized methods of computing has not been fully explored.

The central-difference formula subsequently referred to offers an unusually accurate method of determining the definite integral but at the same time presents practical difficulties in the calculating and handling of differences through the seventh order. The following discussion, with corresponding examples, will show two ways of overcoming the practical difficulties involved.

A simple, accurate, flexible method of determining the value of the definite integral may be described as follows:

1. Determine the ordinates between and including the specified limits of integration, maintaining a constant interval.

2. Multiply the *first* nine ordinates respectively by the following nine factors:

0.3338047013
1.328926091
0.6872208443
1.261894566
0.8333333333
1.071438767
0.9794458223
1.004407242
0.9995286325

3. Multiply the *last* nine ordinates respectively by the same factors in reverse order.

4. Add all the eighteen products formed in the above steps to the unmultiplied remaining ordinates.

5. Multiply this sum by the constant interval; the resulting product is the definite integral between the originally specified limits of integration.

Example: Find the value of $\int_{.4}^{.9} X^7\, dX$

X		*Ordinate*		*Factor*	*Product*
.400		.001638400000		.3338047013	.0005469056
.425		.002504508502		1.328926091	.0033283067
.450		.003736694531		.6872208443	.0025679344
.475		.005455760125		1.261894566	.0068845941
.500		.007812500000	(Step 2)	.8333333333	.0065104167
.525		.01099297205		1.071438767	.0117782964
.550		.01522435234		.9794458223	.0149114283
.575		.02078140531		1.004407242	.0208729940
.600		.02799360000		.9995286325	.0279804047
.625		.03725290298			.0372529030
.650	(Step 1)	.04902227891			.0490222789
.675		.06384492921			.0638449292
.700		.08235430000		.9995286325	.0823154809
.725		.1052848896		1.004407242	.1057489056
.750		.1334838867		.9794458223	.1307402352
.775		.1679236701		1.071438767	.1799199300
.800		.2097152000	(Step 3)	.8333333333	.1747626667
.825		.2601223326		1.261894566	.3282469580
.850		.3205770883		.6872208443	.2203072573
.875		.3926959038		1.328926091	.5218638324
.900		.4782969000		.3338047013	.1596577538
					2.149064412 *(Step 4)*

Step 5. The sum in Step 4 multiplied by the constant interval (.025) of the example equals .05372661030 which is the value of the integral $\int_{.4}^{.9} X^7\, dX$, with an error of only .00000012905.

In the event that four *extra* ordinates can be accurately determined at each end of the original group of ordinates, the following even more accurate method is applicable:

1. Determine the ordinates between and including the specified limits of integration, maintaining a constant interval; then determine *four* more ordinates at each end of the group, still maintaining the constant interval.

2. Multiply the *first* nine ordinates of the *entire* group respectively by the following nine factors:

0.0004713679453
−0.004407242063
0.02055417763
−0.07143876764
0.5
1.071438767
0.9794458223
1.004407242
0.9995286325

3. Multiply the *last* nine ordinates of the entire group respectively by the same factors in reverse order.

4. Combine all eighteen products with the unmultiplied remaining ordinates.

5. Multiply the above sum by the constant interval; the resulting product is the definite integral between the originally specified limits of integration.

Example: Find the value of $\int_{.4}^{.9} X^7 \, dX$

X	Ordinate	Factor	Product
.300	.0002187000000	.0004713679453	.0000001031
.325	.0003829865538	− .004407242063	−.0000016879
.350	.0006433929688	.02055417763	.0000132244
.375	.001042842865	− .07143876764	−.0000744994
.400	.001638400000	.5	.0008192000
.425	.002504508502	1.071438767	.0026834275
.450	.003736694531	.9794458223	.0036598898
.475	.005455760125	1.004407242	.0054798050
.500	.007812500000	.9995286325	.0078088174
.525	.01099297205		.0109929721
.550	.01522435234		.0152243523
.575	.02078140531		.0207814053
.600	.02799360000		.0279936000
.625	.03725290298		.0372529030
.650	.04902227891		.0490222789
.675	.06384492921		.0638449292
.700	.08235430000		.0823543000
.725	.1052848896		.1052848896
.750	.1334838867		.1334838867
.775	.1679236701		.1679236701
.800	.2097152000	.9995286325	.2096163471
.825	.2601223326	1.004407242	.2612687547
.850	.3205770883	.9794458223	.3139878899
.875	.3926959038	1.071438767	.4207496150
.900	.4782969000	.5	2391484500
.925	.5794181954	− .07143876764	−.0413929218
.950	.6983372961	.02055417763	.0143537488
.975	.8375915935	− .004407242063	−.0036914689
1.000	1.000000000	.0004713679453	.0004713679
			2.149059250 *(Step 4)*

(Step 1) (Step 2) (Step 3)

Step 5. The result of Step 4 multiplied by the constant interval (.025) of the example equals .05372648125 and is the correct value of the original integral, $\int_{.4}^{.9} X^7 \, dX$, to the number of places shown.

In either of the foregoing numerical integration systems the number of ordinates chosen may be either even or odd. Also, advantage is gained in an extended calculation since the majority of the ordinates are simply added without change. Further, there is no limit to the number of ordinates that can be used although only nine at each end of the group are multiplied by factors.

In view of these advantages, the establishment of two permanent factor files of only eighteen cards each is all that is necessary to make both systems readily adaptable to punched card equipment.

The two preceding systems of integration are based on a central difference formula carried through the fifth differences by Scarborough.[1, 2] The author is indebted to Dr. J. W. Wrinch of the David W. Taylor Model Basin for deriving an additional term containing the seventh differences.[3]

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DISCUSSION

Dr. Blanch: These factors are related to those in our volume of Lagrangian coefficients.[1]

Dr. Grosch: It is possible to generate integration coefficients of this sort based on other than polynomial approximation. Unfortunately in most cases the weighting factors become functions of the limits of integration, but if the latter are fixed for a large group of problems one may find it economical to derive factors based on an approximating function that suits the physical situation.

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Integration of Second Order Linear Differential Equations on the Type 602 Calculating Punch

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THIS PAPER is concerned with the numerical solution by means of punched cards of the equation $\frac{d^2S}{dx^2} = g(x)S + p(x)$ with initial value conditions. In the second section, the special case of $p = 0$ is treated. By approximation to a difference equation, a step-by-step procedure is derived which has been set up for automatic integration on the Type 602. The machine operation is described in the third section. In the fourth section, the method is extended to the general second-order linear differential equation which can be reduced to the case above, where $p \neq 0$. This can be set up on the Type 602 with a slight change of the control panel described in the third section.

Approximation by Finite Differences

Consider the second-order linear differential equation

$$\frac{d^2S}{dx^2} = g(x)S \tag{1}$$

where g is a given function over the integration interval.

Suppose that in the neighborhood of an arbitrary point x_n of the integration interval, a solution S and its second derivative can be developed into a Taylor series. Let the constant step in x be h. The value of the function S at the point $x_{n+1} = x_n + h$, is then

$$S_{n+1} = S_n + hS_n^{\text{i}} + \frac{h^2}{2!}S_n^{\text{ii}} + \frac{h^3}{3!}S_n^{\text{iii}} + \frac{h^4}{4!}S_n^{\text{iv}} + \frac{h^5}{5!}S_n^{\text{v}} + \ldots \tag{2}$$

Here $S_n^{\text{i}} = \left(\frac{dS}{dx}\right)_{x_n}$ and so forth.

The same development of the second derivative gives

$$S_{n+1}^{\text{ii}} = S_n^{\text{ii}} + hS_n^{\text{iii}} + \frac{h^2}{2!}S_n^{\text{iv}} + \frac{h^3}{3!}S_n^{\text{v}} + \ldots \tag{3}$$

In order to eliminate the fourth derivatives, a modified function y can be defined

$$y = S - \frac{h^2}{12}S^{\text{ii}} \tag{4}$$

Insertion of (2) and (3) gives the value of y at the point x_{n+1}

$$y_{n+1} = S_n + hS_n^{\text{i}} + \frac{5h^2}{12}S_n^{\text{ii}} + \frac{h^3}{12}S_n^{\text{iii}} - \frac{h^5}{180}S_n^{\text{v}} - \frac{h^6}{480}S_n^{\text{vi}} + \ldots \tag{5}$$

and by reversing the sign of h

$$y_{n-1} = S_n - hS_n^{\text{i}} + \frac{5h^2}{12}S_n^{\text{ii}} - \frac{h^3}{12}S_n^{\text{iii}} + \frac{h^5}{180}S_n^{\text{v}} - \frac{h^6}{480}S_n^{\text{vi}} + \ldots \tag{6}$$

The central difference of y around the point x_n is now

$$\delta^2 y_n = y_{n+1} - 2y_n + y_{n-1} \; . \tag{7}$$

Insertion of (5) and (6) yields

$$\delta^2 y_n = h^2 S_n^{\text{ii}} - \frac{h^6}{240}S_n^{\text{vi}} + \ldots \tag{8}$$

Terms of the sixth and higher orders are going to be neglected and thus form the truncation error. The approximation gives

$$\delta^2 y_n = h^2 S_n^{\text{ii}} \; . \tag{9}$$

Insertion of (1) in (9) and (4) yields

$$\delta^2 y_n = g_n h^2 S_n \tag{10}$$

and

$$y_n = (1 - g_n h^2/12)\, S_n \; . \tag{11}$$

After rewriting equation (11)

$$S_n = (1 + \mu_n)\, y_n \tag{12}$$

where

$$\mu_n = \frac{g_n h^2/12}{1 - g_n h^2/12} \; . \tag{13}$$

Now the approximation of the difference equation (10) will be

$$\delta^2 y_n = \gamma_n y_n \quad (14)$$

with

$$\gamma_n = 12\mu_n \ . \quad (15)$$

The derivation given here* is essentially the method used in a paper by Feinstein and Schwarzschild.[1] The main machine used in their work was a special multiplying punch.

In order to apply equation (15) for computation on the Type 602, it will be written in a different manner. With the advancing difference notation

$$\Delta y_n = y_{n+1} - y_n \ , \quad (16)$$

equation (14) is

$$\Delta y_n = \Delta y_{n-1} + \gamma_n y_n \ . \quad (17)$$

The initial values are y_1 and Δy_0. In (17), putting $n = 1$,

$$\Delta y_1 = \Delta y_0 + \gamma_1 y_1 \ . \quad (18)$$

From (17) it is evident that every difference Δy_n is computed from the previous one by adding $\gamma_n y_n$. This procedure will give

$$\Delta y_n = \Delta y_0 + \sum_{i=1}^{n} \gamma_i y_i \ . \quad (19)$$

After having used (19), the next value of y is computed from (17)

$$y_{n+1} = y_n + \Delta y_n \ . \quad (20)$$

The Machine Set-up

In order to describe how the equations (12), (19) and (20) are set up for automatic integration on the Type 602, a flow chart will be used. In this, the first eight columns represent the eight counters MC, MP, RHC, LHC, Summary Counters 13, 14, 15 and 16. The chart is divided into ten rows, indicating machine cycles, which are shown in the ninth column.

Suppose that the integration has proceeded up to the nth integration cycle (not to be confused with machine cycles). This means that $n-1$ cards have been run through the machine, each one of them carrying the functions γ and μ. Every card carries the integration one step forward and is punched with the computed values of y and S.

The machine has computed y_n during the $(n-1)$th integration cycle and keeps the absolute value of this number stored in the MP counter during the reading of the nth card. In order to perform its function correctly, MP must contain a true number. The sign of y_n has to be stored elsewhere, for which purpose SC 13 is reserved. If the contents of this counter are zero or a true number, it is interpreted as a positive sign in y_n. If the contents are a complement number, it means that y_n is less than zero.

Moreover, the machine stores the progressive differences of y in SC 14 and 16 which are coupled together. In the beginning of the nth integration cycle, the stored difference is

$$\Delta y_{n-1} = \Delta y_0 + \sum_{i=1}^{n-1} \gamma_i y_i \quad (21)$$

as indicated in the flow chart (Figure 1, page 36).

The essence of the procedure is the following. During the nth card operation, the machine is going to compute the product $\gamma_n y_n$ and add it to Δy_{n-1}, thereby yielding Δy_n, according to equation (19); Δy_n is added to y_n, giving y_{n+1}; y_n is multiplied with $1 + \mu_n$ which gives S_n by (20) and (12). Finally S_n and y_{n+1} are punched out on the nth card.

The details of the operation appear in the flow chart. During the card reading cycle, the precomputed numbers γ_n and $1 + \mu_n$ are fed from the card into MC and SC 15. The numbers in the MC and MP counters are then multiplied. The rounded product comes out in LHC. This product is transferred to SC 14 and 16 on the first program cycle. As it appears positive in LHC, the sign has to be taken care of by adding the product into SC 14 and 16 if it is positive, and subtracting it, if negative. This is accomplished in the following manner: an eventual sign in γ_n appears as an X punch in the card, wired to pick up an entry control selector. A negative sign in y_n appears as previously mentioned in SC 13 which therefore is balance-tested during the first program cycle. If y_n is negative, the test impulse will get through the balance control. From there it is wired to pick up a selector. By coupling the above-mentioned selectors in series, a plus or a minus shot will be available from them, due to the sign of the product. The impulses are then fed into the appropriate control hubs of SC 14 and 16 which will take care of entering the product into the counters with the correct sign. The details of the wiring are shown in the control panel diagram.

The quantity $1 + \mu_n$ is transferred into MC on the second program cycle and multiplied with y_n during the third program cycle. The product S_n will appear in LHC where it is stored until the transfer-to-storage cycle. The sign is taken care of by balance-testing SC 13, as the sign of S_n is always the same as that of y_n. That sign will now be stored in LHC; SC 13 can be used for other purposes and is reset on the fourth program cycle.

On the fifth program cycle, MC reads out y_n to SC 13 and 15, coupled together. MC is reset on the same cycle. As MC always stores true numbers, the sign is taken care of by balance-testing LHC, picking up a selector and feed-

*From lectures by Dr. L. H. Thomas.

MC	MP	RHC	LHC	SC 13	SC 14	SC 15	SC 16	SEQUENCE
γ_n	$\lvert y_n \rvert$ $\{y_1\}$			Contains sign of y_n	$\Delta y + \sum_{i=1}^{n-1} \gamma_i y_i$ $\{\Delta y_0\}$	$1 + \mu_n$	$\Delta y_0 + \sum_{i=1}^{n-1} \gamma_i y_i$ $\{\Delta y_0\}$	Read card
RC		RC	$\gamma_n y_n$					Multiply
			RO to SC 14 & 16, RC Sign Ctrl. by Bal. Test of SC 13 and Entry Ctrl. Sel. 2	Balance Test	y_n		y_n	1st Program Cycle
$1 + \mu_n$						RO to MC		2nd Program Cycle
		RC	$(1 + \mu_n)\, y_n = S_n$	Balance Test				3rd Program Cycle
				RC		RC		4th Program Cycle
	RO to SC 13 & 15, RC		Balance Test	y_n		y_n		5th Program Cycle
				$y_n + \Delta y_n$	RO to SC 13	$y_n + \Delta y_n$	RO to SC 15	6th Program Cycle
	$\lvert y_{n+1} \rvert$			RO to MP Balance Test		RO to MP		7th Program Cycle
			RO S_n to Storage, RC	RO y_{n+1} to Storage		RO y_{n+1} to Storage RC		Transfer to Storage Cycle
γ_{n+1}						$1 + \mu_{n+1}$		Read Next Card

FIGURE 1

ing an appropriate impulse to the plus or minus hubs of SC 13 and 15. As distinguished from MC and MP, the summary counters and LHC are used to store negative numbers as complements.

On the sixth program cycle, Δy_n is read out from SC 14 and 16 and accumulated in SC 13 and 15. This means that the operation (20) is carried out.

During the seventh program cycle, y_{n+1} is read into MP. For reasons previously mentioned, it has to be subtracted into MP if negative or added if positive. This is again taken care of by balance-testing SC 13.

The program cycles are now finished and the machine transfers to storage. S_n is read out and punched from LHC, y_{n+1} from SC 13 and 15. By using the balance punch feature, negative signs will appear as X punches.

LHC and SC 15 are reset, as distinguished from SC 13 which has to store the sign of y_{n+1} for the next card. If the sign is negative, there will be a nine in the left-most position of SC 13 that will be detected during the balance tests. The allowed number of digits in the function y must be limited to nine as there are ten positions in SC 13 and 15 together and the left position has to be left free.

The initial values y_1 and Δy_0 are fed into MP and SC 14 and 16 from the first card. In the flow chart they appear within braces. As all counters are reset before the starting of a new integration, the zeros in SC 13 will indicate the positive sign of the initial value y_1.

The control panel has been wired as shown in the wiring diagram (Figure 2) and the procedure has been found to work satisfactorily. The coefficients of integration, γ and $1 + \mu$, are conveniently computed with punched card machines. In the mentioned set-up μ was calculated in one run of the 602 ; γ and $1 + \mu$ came out together in one run on the 601.

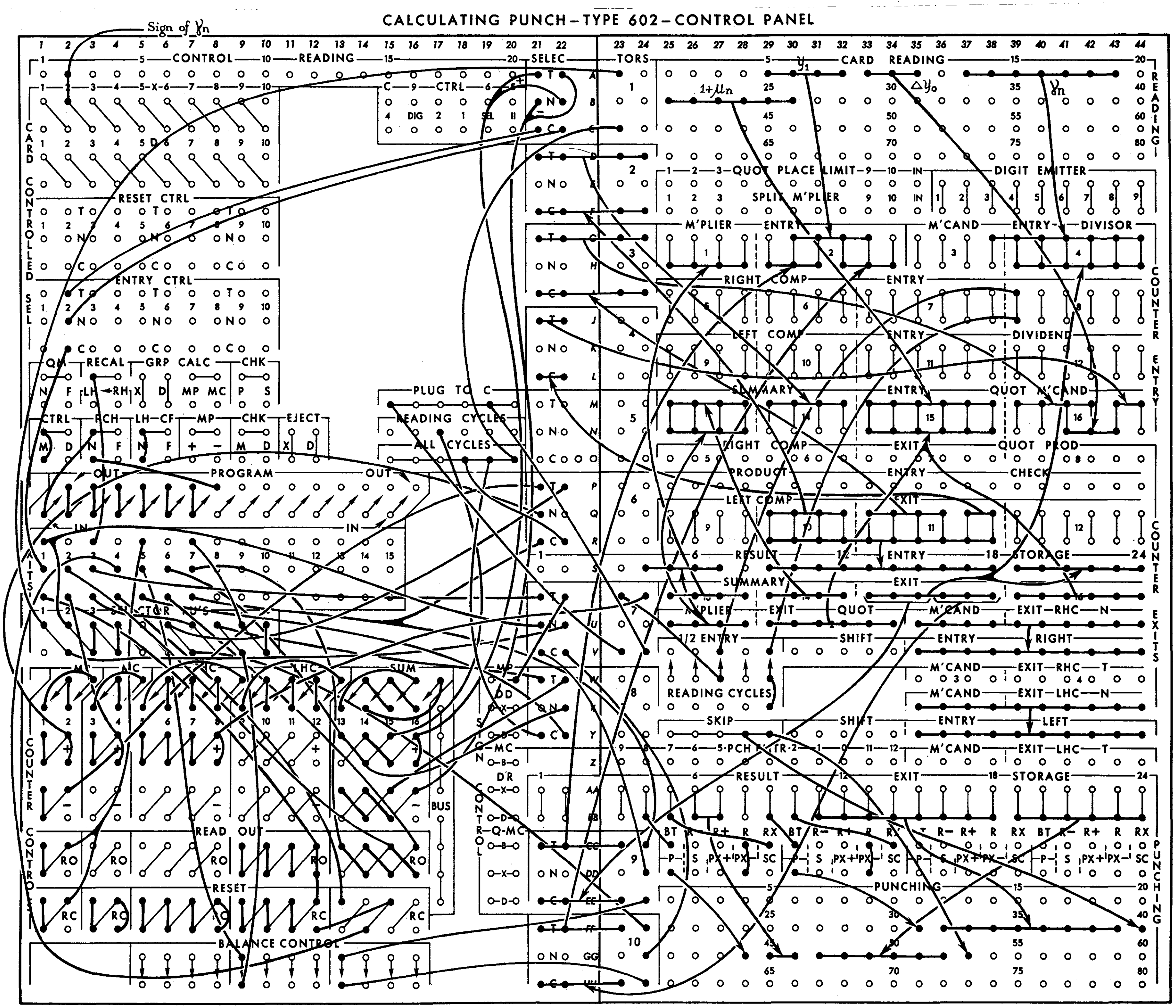

CALCULATING PUNCH—TYPE 602—CONTROL PANEL
Sign of γ_n
y_1
$1+\mu_n$
Δy_0
γ_n

FIGURE 2

The General Case

Any linear second-order differential equation can be reduced to the form[2]

$$\frac{d^2S}{dx^2} = g(x)\,S + p(x) \qquad (22)$$

where the first derivative is lacking. This equation, inserted in (9) and (4), will give recurrence formulas corresponding to (12), (19) and (20):

$$S_n = (1 + \mu_n)\,y_n + q_n/12 \qquad (23)$$

$$\Delta y_n = \Delta y_0 + \sum_{i=1}^{n}(\gamma_i y_i + q_i) \qquad (24)$$

$$y_{n+1} = y_n + \Delta y_n\ . \qquad (20)$$

The addition of $q_n = h^2\,p_n\,(1 + \mu_n)$ in each step is what makes (24) differ from (19). It can be done in the setup of Section 3 by feeding q_n from the nth card into SC 14 and 16 which accumulate the difference Δy_{n-1}.

As for (23), there is no counter storage left for the additional term $q_n/12$. This can be overcome, however, by letting the machine compute $(1 + \mu_n)\,y_n$ in the integration run and then add the above-mentioned term in an extra run. The calculation of S can be split in this manner because it is not part of the progressive computation of y; q_n and $q_n/12$ can be computed in one run on the 602.

Another way of relieving the situation is the following. Define an auxiliary function

$$z = y + h^2\,p/12\ . \qquad (25)$$

After insertion of z, the equations (23), (24) and (20) will be

$$S_n = (1 + \mu_n)\,z_n \qquad (26)$$

$$\Delta z_n = \Delta z_0 + \sum_{i=1}^{n}(\gamma_i z_i + P_i) \qquad (27)$$

$$z_{n+1} = z_n + \Delta z_n\ . \qquad (28)$$

Here

$$P = h^2\,(p + \delta^2 p/12)\ . \qquad (29)$$

Equations (26) and (28) are basically the same as (12) and (20) of Section 2. Equation (27) is taken care of in the same manner as (24), by feeding the additional term P_n from the nth card into the counters that accumulate the difference in z. As P can be precomputed in one run on the 602, the extra run of equation (23) is saved.

Thus it has been shown that the general second-order linear differential equation can be automatically integrated on the 602. Reduction to the form (22) has not been treated here. It will probably imply the same amount of work as the rest of the procedure.

REFERENCES

1. L. FEINSTEIN and M. SCHWARZSCHILD, "Automatic Integration of Linear Second-Order Differential Equations by Means of Punched Card Machines," *Rev. Sci. Inst.*, 12 (1941), pp. 405-8.
2. *Ibid.*, p. 405.

DISCUSSION

[This paper and the following one by Dr. Paul Herget were discussed as a unit.]

Integration of the Differential Equation $\frac{d^2P}{dr^2} = P \cdot F(r)$ *Using the Type 601 Multiplying Punch*

PAUL HERGET

Cincinnati Observatory

THE EQUATION $\frac{d^2P}{dr^2} = P \cdot F$ (where F is a predetermined function of r), arises in the computation of the wave equations for atoms in various stages of ionization. In that case it is necessary to replace F by $(F + E)$, where E is such a constant as will cause P to vanish when r approaches infinity. In practice the correct value of E must be determined by trials, and hence it is necessary to run through this kind of a solution many times.

From the calculus of finite differences we have the following relationships:

$$(\Delta r)^2 \frac{d^2P}{dr^2} = (\Delta r)^2 P\,(F + E) = f$$

$$\Delta^{ii}P = f + \Delta^{ii}f/12 - \Delta^{iv}f/240 + \ldots$$

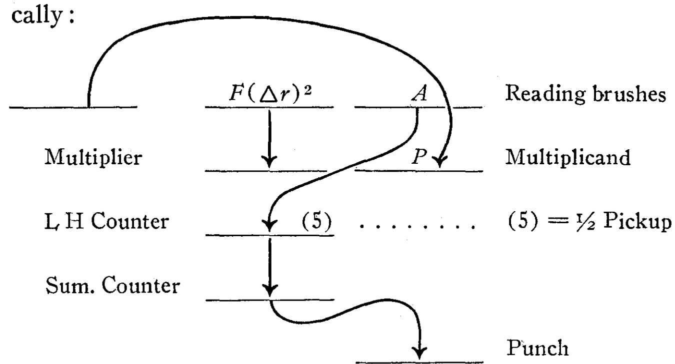
The ultimate objective of our computations is to obtain a table of numerical values of $P(r)$ which satisfy these two conditions and which may be illustrated by the following arrangement of the intermediate results:

r	P	$\Delta^{i}P$	$\Delta^{ii}P$	f	$\Delta^{i}f$	$\Delta^{ii}f$	$\Delta^{iii}f$
0.0	1.000 000		−0.000 012	0.000 000		−146	
		−0.072 712			+927		+ 3
0.1	0.927 288		+0.000 915	0.000 927		−143	
		−0.071 797			+784		+ 4
0.2	0.855 491		+0.001 699	0.001 711		−139	
		−0.070 098			+645		+ 9
0.3	0.785 393		+0.002 345	0.002 356		−130	
		−0.067 753			+515		+ 8
0.4	0.717 640		+0.002 861	0.002 871		−122	
		−0.064 892			+393		+12
0.5	0.625 748		+0.003 255	0.003 255		−110	

This illustration represents the solution of the simplified equation $\frac{d^2P}{dr^2} = Pr$, where $P(0) = 1$, and $P(\infty) = 0$; the problem is to find dP/dr at $r = 0$ such that the condition $P(\infty) = 0$ will be fulfilled.

The only numbers which can be entered directly into the table are in the f column, when they are computed according to the first equation above. When these are differenced, it becomes possible to compute $\Delta^{ii}P$ and build up the $\Delta^{i}P$ and P columns by addition. It is necessary to proceed step by step and by successive approximations.

In the solution of the original, more general, equation it was possible to employ a Type 601 Multiplying Punch equipped with sign control and a net balance summary counter. The board wiring may be illustrated schematically:



The first position in the punched field receives an X punch. The group multiplier switch is wired OFF and ON at the same time. The OFF switch permits the multiplier to reset on every card. The column in which the X is punched is wired to read as if it were the group multiplier master card indication. This has the effect that when any card is punched and then fed through the machine a second time it will be skipped out as a master card, without punching.

The field A is always crossfooted into the LHC and it transfers to the SC with sign control. The only multiplier is $(\Delta r)^2F$, which is prepunched into a set of salmon cards. The multiplicand is wired reversed from the positions where the punched field is "reflected" through the center of the card. All cards have their index numbers prepunched. The P's are manila cards, the $\Delta^{i}P$'s are green cards, and the $\Delta^{ii}f/12$'s are blue cards. These must be obtained from a previous approximation and have the above mentioned X punch.

One cycle of operations consists of the following (the index i refers to the ith *horizontal* line in the numerical table):

Card Color	*Machine Operation*	*Result in Sum. Counter*
Salmon (i)	Multiply F by $P(i)$	Punch f
Blue (i)	Add $\Delta^{ii}f/12$	Non punch $\Delta^{ii}P(i)$
Green $(i-1/2)$	Add $\Delta^{i}P(i-1/2)$	Non punch $\Delta^{i}P(i+1/2)$
Green $(i+1/2)$	Blank	Punch $\Delta^{i}P(i+1/2)$
Manila (i)	Add $P(i)$	Non punch $P(i+1)$
Manila $(i+1)$	Blank	Punch $P(i+1)$
Salmon $(i+1)$	Blank and reversed	Punch $P(i+1)$

The operator has the cards of different colors piled separately before him, each pile in order of the index i. On one cycle he performs the following sequence of operations:

1. The salmon card is allowed to fall into the stacker.
2. The blue card is allowed to fall into the stacker.
3. The top card from the blue pile is picked up and held in one hand.
4. The first green card is allowed to fall into the stacker.
5. The second green card is placed behind the blue card being held in one hand.
6. The top card from the green pile is picked up and placed behind the other cards being held in one hand.
7. The first manila card is allowed to fall into the stacker.
8. The second manila card is placed behind the other cards being held in one hand.
9. The top card from the manila pile is picked up and placed behind the other cards being held in one hand.
10. The top card from the salmon pile is picked up and placed in the *reversed* position behind the other cards being held in one hand.
11. The last salmon card is placed (in the direct position) *ahead* of all the cards being held in one hand, and
12. This deck is now placed in the feed hopper to begin the next cycle.

The operator may be illiterate, so long as he is not color blind! The work proceeds at the rate of thirty seconds per step in the table, which is nearly the speed at which the cards can pass through the machine.

The SC does not reset except under control of the class selector. The selector transfer is obtained only from a pre-punched X on the salmon card when it is fed in the reversed position. This automatically clears the counter at the end of each cycle of operations.

If we undertake to apply these principles to the solution of the first order differential equation, $(\Delta r)\frac{dP}{dr} = f$, it works out as follows:

$$P(i) = 'f(i) - \Delta^{i}f(i)/12 + 11\,\Delta^{iii}f(i)/720 - \ldots$$

where it will be noted that all the quantities on the right side are "on the line" in odd difference columns, so that they are actually the means of the quantities on the half lines above and below. Then

$$\Delta^{i}P(i+1/2) = \frac{1}{2}f(i) - \frac{1}{24}\Delta^{ii}f(i) + \frac{11}{1440}\Delta^{iv}f(i)\ldots$$

$$+\frac{1}{2}f(i+1) - \frac{1}{24}\Delta^{ii}f(i+1) + \frac{11}{1440}\Delta^{iv}f(i+1)\ldots$$

Now, f may consist of the algebraic sum of any number of cards, and if the higher order difference cards are already available from a previous approximation, it is only necessary to include one control card in each control group and to use two counters in order to build up the table of numerical values of P. All the cards representing quantities on the ith line are entered into both counters. The control change causes an intermediate (progressive) total. This gives the value of the integral, $P(i)$, on the ith line. As the next control group starts through the machine, the first card is the control card, and this rolls the second counter into the first counter, then causes a minor total which clears the second counter. This enters all of the quantities from the ith line which are needed for $\Delta^{i}P(i+1/2)$ into the first counter. Then the remaining cards in that control group enter the quantities from the $(i+1)$th line into both counters, as before, and the resultant progressive total in the first counter is $P(i+1)$.

DISCUSSION

Dr. Grosch: I would like to make the general remark that Dr. Herget has a big point in using the human element in his cycle. We can all make use of the tricks that Mr. Bell and Dr. Herget have described. If you reduce the number of 601 or 602 control panels that are kept wired up by thirty or forty percent, you have effected a substantial saving. We did an optical calculation two years ago which required twenty-eight boards. I don't think we had heard of the reversed card at that time. That one idea would have released eight 601 control panels for other work.

Dr. Stanley: I have one question. Notice that in Mr. Lindberger's equation (22)

$$\frac{d^2S}{dx^2} = g(x)\,S + p(x)$$

$g(x)$ is a function of one variable only. The equation is linear and of the second order. Now a more general equation of the second order in the normal form may be written

$$\frac{d^2S}{dx^2} = g(x, S)\,S + p(x)$$

This equation is non-linear. You might conceivably treat it in the same manner as the speaker has suggested, except that you would obviously run into the difficulty of computing the quanties γ_n. We might construct some suitable program beforehand and use it to estimate the γ_n. I wonder if either of you gentlemen have tried such a method.

Dr. Thomas: This is exactly the thing that Hartree did in his so-called "Self-consistent Field" computations. There are two ways you can do it. With the notation:

$$-\frac{h^2}{8\,\pi^2 m}\nabla^2 \psi_n + V_n \psi_n = E_n \psi n$$

where

$$V_{n1} = V_1 - e\int \frac{\psi_{n2}^2}{r_{12}} dr_2$$

He used V and any numerical approximation to ψ_n to get V_n, then solved the differential equation to get ψ_n, these to get new V and V_n, and so on until you come out with what you put in. A somewhat different trick was one we tried a few years ago. Instead of assuming V_n we assumed V and put ψ_n to get V_n continuously as ψ_n was being computed. I don't think you gain anything by that, except that every answer you get is a solution of the differential equation.

Dr. Caldwell: It might be possible to do that kind of thing with the 602 provided the functions were not too complicated.

Dr. Thomas: It was the double integral that I had in mind. You can do two of them simultaneously on the 405 as well as the constant. You go all through an integration to get preliminary values for ψ_n. These must be normalized. These integrals must be obtained to get an "energy" to put in on the right-hand side before repeating the integration.

Some Elementary Machine Problems in the Sampling Work of the Census

A. ROSS ECKLER

Bureau of the Census

PERHAPS I have a definite advantage over most of you in being able to recognize the value of this kind of meeting. I do not come here as a mathematician nor as an expert in machine accounting; so perhaps I am peculiarly able to see the advantages of bringing together these two types of people. In my opinion the International Business Machines Corporation is to be commended for its vision in making possible meetings of this kind. The advantages for both groups are very great, and I have been much impressed with the gains from this sort of meeting even though much of the material is highly technical.

Most of you are familiar with the long-run interest of the Census in large scale accounting equipment. We are very proud of the fact that in the early years men like Hollerith and Powers were employees of the Bureau of the Census, and we have for many years used equipment specially developed for our needs. We have used that as well as very large quantities of the different types of IBM equipment.

I shall speak primarily of our work in the field of sampling, which involves certain applications of equipment somewhat different from what we get in our complete tabulations, and which illustrates some areas in which the present equipment fails to meet the requirements that we would like to see met.

It is unnecessary to inform this group about the advantages of sampling. Most of you are familiar with the theoretical work to a far greater degree than I am. You doubtless know that through the application of sampling we have been able to save very large sums of money in our tabulating work. Moreover, we have been able to speed up results so that we have been able to carry out many types of detailed tabulations which would be far too expensive to carry out on the basis of complete coverage.

There are several directions in which we apply sampling. One is the use of sampling to serve as a supplement to a complete census, asking certain questions on a sample basis only. In this way, we have been able to increase very greatly the number of subjects covered.

The second way in which we use sampling is to carry out independent field surveys based upon a sample of the population from which we can estimate the total population of the country and the population in various economic and social groups.

The third way in which we use sampling is in connection with measuring or controlling the quality of statistical operations. I will refer to each of these uses very briefly in some of the applications I will mention.

First of all, I should like to refer to an application of the machines which is a very happy one. This use is in connection with drawing samples of blocks for certain types of surveys. We want to determine certain blocks in which we are going to collect information. We have put in punched cards certain facts relating to each block in all of our cities. That information, among other subjects, includes the number of dwelling units, the number of stores, and the number of various types of institutions. As we take our population sample, we want to select certain blocks in which we will do our sampling.

We have determined that under many circumstances an efficient procedure of drawing the sample of blocks is to draw it on the basis of probability proportionate to size, i.e., the number of dwelling units, or number of stores. We have been able to develop a procedure for selecting blocks by the use of the Type 405 whereby we run through the cards, accumulate, and select every *n*th dwelling unit. The machine can be wired so that if in a very large block there are two or more units which are to be included in the sample, this fact will be indicated by the machine. If any of you are interested in that, we would be glad to have you write to inquire about the method.

Another area in which we have made use of sampling is in connection with the processing of data. We are particularly interested in the development of better equipment to handle sample materials because it will give us a possibility of increasing the use of sampling, thereby taking greater advantage of the benefits it offers. We are anxious to extend the use of this tool as far as possible, and in certain areas the mechanical equipment is a limiting factor. We could go further with it if we had equipment which fitted the needs more precisely than the present equipment does.

Just as we depend upon equipment to expand the use of sampling, we also use sampling to improve the use of equipment. We are carrying out a great many of our processing operations on the basis of sample verification. This takes place in a great many fields; one example is our foreign trade statistics, which involve tabulations of information on imports and exports by country of origin, country of destination, etc. We have developed a system of sample verification, which usually provides for a sample of one card in fifty. We continue with that sample as long as the operator is making fewer than sixteen errors per 400 cards sample verified. When she exceeds that rate of error, we shift over to 100 per cent verification for a short time. Then, when the evidence is available to show that the person has come back down to a lower error rate, we shift back to a five per cent sample and after a period of that, if the rate still continues low, we go back to the two per cent sample.

Through the use of that type of sample operation, we have been able to have the verification of the work of thirty operators handled by three. This achieves a very considerable saving in the verification operation, and still provides control of the work so that we can be sure that our error rate is under two per cent.

Now I should like to mention the major applications of sampling which take place in a number of fields in the Bureau. In the field of current population surveys we are carrying out samples on a monthly basis. We are doing somewhat the same kind of work in the field of business statistics for retail and service firms, and generally similar work in government statistics, where we collect employment data for state and local government units.

In the first field I mentioned, our current population surveys, we interview a sample of about 25,000 households once a month. We get information from them on the number of people who are employed, the number unemployed, the hours worked, the occupation, industry, and so forth. The households are selected by the use of area sampling, a method probably familiar to most of the people in this room. It is based upon units which are selected from sixty-eight different sample areas scattered around the country, scientifically determined so as to give a good cross section of the country as a whole. We insist that all of our samples have measurable accuracy; in other words, that the design be such that we can determine the degree of error in the results.

In this current survey of 25,000 households we estimate that the figure on the total labor force will be within one per cent of that which would be obtained from a complete census nineteen times out of twenty. We achieve that very high degree of accuracy partly by virtue of the fact we have control totals for various groups to which we can adjust the sample results. Obviously, in a sample survey of this sort giving monthly information, speed is of greatest importance. These data are highly perishable and it is important we make them available as rapidly as possible because they are widely used. The information we get for these 25,000 households is punched in about 65,000 cards for individuals and those cards are weighted according to the sampling ratios that were used. As each card gets a weight which depends upon the age, sex, and residence group of the population from which it is drawn, a considerable number of weights must be applied. There is nothing about that job which can not be handled by standard equipment. The difficulty is that it takes quite a while to carry it out.

It is necessary first of all to sort the cards into these sub-groups, to determine the total number in each, and then to determine the weight which has to be applied to each type of card. Even after considerable experience, we found ourselves unable to do the whole job in less than about thirty days, which meant that the data collected in one month would appear fairly late in the next month.

After some experimental work we shifted over to a Census-built machine, the unit counter, which has the advantage of somewhat greater speed of operation; for this particular kind of job, it can count in a considerable number of categories—sixty—at the rate of 400 per minute.

This procedure has certain disadvantages, however; in order to use this machine we are forced to use a less precise weighting system than we could use on the 405. We accomplish the weighting by classifying our cards in 144 different groups and then rejecting some cards by random methods and duplicating some others by random methods so as to get our results weighted according to the predetermined weights.

With that slight sacrifice of precision we were able to speed up results a great deal and I believe, at the present time, we use about fifteen days. I think it is pretty clear we don't yet have ideal equipment for meeting this particular problem.

One other area of work I will mention briefly in closing. In connection with our sample work, we attempt to establish a very careful measure of the degree of accuracy of the results so that we have to compute large numbers of variances and, as you well know, that involves calculating very large numbers of sums of squares and sums of products. In fact, for the measurement of accuracy of just one item, it is necessary to get the squares of more than three thousand numbers, to weight them by certain factors, and then to combine them. It is possible to get these sums of squares and sums of products through a rather complicated series of operations, but the time required for that is considerable. It is not a very efficient procedure. Some consideration has been given to the extent to which some of the new high-speed multipliers will meet the problem, but study of the situation so far indicates that we are still considerably handicapped in the direction of getting these measures of the accuracy of the results of sample data. There is a need for further development which will increase the use of sampling by making it possible to measure the accuracy of the results more speedily.

I have brought several types of exhibits in which some of you may be interested, a number of pamphlets and bulletins which show cases in which we have made direct use of machine tabulation sheets for publication: some of our housing reports, some of our foreign trade reports, and a job we did for the Air Forces, on all of which we printed the 405 sheets directly. If there are questions you want to ask about them, feel free to write in. There are also several copies of the booklet on work of the Census Bureau—*Fact-Finder for the Nation*. If any of you want a copy of that, I would be very glad to furnish it.

DISCUSSION

Mr. Tillitt: At one time the Bureau of the Census turned out a little sheet called "Tab Tips." Is that distributed any more?

Mr. Eckler: I think I have the first forty. If anything of that sort is being distributed now, it has not come to my attention.

IBM Applications in Industrial Statistics

CUTHBERT C. HURD

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THE NUMBER of IBM applications in Oak Ridge is great, partly because of the variety of activities in which Carbide and Carbon Chemicals Corporation engages there, and partly because of the fundamental importance of methods of probability and of mathematical statistics in the atomic energy field. Thus, on the one hand, Carbide and Carbon, an Atomic Energy Commission contractor, operates the Oak Ridge National Laboratory, the Magnetic Separation Plant, and the Gaseous Diffusion Plant, the first of these being devoted to fundamental research, the second and third to development and to production. On the other hand, atomic energy problems in whatever state of development are problems which require the most careful kind of statistical analysis both in the design of experiments, in the interpretation of experimental results, and in maintaining statistical quality control in various production and measurement programs. When the size of a problem, as measured by the number of measurements to be made, has been great, we have found IBM methods advantageous and in some cases almost indispensable.

Originally I had thought to give a survey of the type of statistical problems encountered in Oak Ridge and to describe the machine methods which are used in their solution. However, in view of the previous papers of this Forum and particularly in view of informal discussions with other participants I have now decided to describe only two types of problems and on one of these I should like to ask your advice. These problems are: one, that of curve fitting; two, calculating approximate solutions to partial differential equations.

The Problem of Curve Fitting

The Straight Line (One Fixed Variate)

The problem of curve fitting is as old as experimental science and it is familiar to each of you in one connection or another. As a beginning, suppose that we are able to measure each of two variates x and y and that we postulate a linear relation between x and y. That is, we assume that there exists a relation of the form

$$y = \alpha x + \beta \ .$$

If the assumption is now made that measurements on x can be made with perfect precision, that is, that x is a fixed variate, we set x at the values $x_1, x_2, \ldots, x_n$, say, and measure the corresponding values of $y: y_1, y_2, \ldots, y_n$. We realize that the set of n value pairs represents only a sample of all possible value pairs and consequently we are confronted with this problem in statistical inference:

On the basis of the n sample values (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), required to estimate the parameters α and β of the equation above. Now, mathematical statistics is a pure science in the sense that conclusions follow inescapably from assumptions; and, in certain fields, investigators believe that a reasonable assumption for the present problem is that of normality of distribution with fixed variance in y corresponding to each fixed value of x. More specifically it is assumed that if x is fixed at x_1, say, and the corresponding y measured repeatedly then these y values will be normally distributed about a mean value given by $\alpha + \beta x_1$, and a variance of σ^2, say. Similarly repeated measurements of y corresponding to fixing x at x_2 would result in a normal distribution about a mean value of $\alpha + \beta x_2$ with the same variance, σ^2.

Under these conditions it is not difficult to show that the optimum estimates $\hat{\alpha}$, $\hat{\beta}$ of α and β are given as solutions of the simultaneous equations

$$n\alpha + \beta \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

$$\alpha \sum_{i=1}^{n} x_i + \beta \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i \ .$$

I will not define optimum, but will only say that in the case considered this procedure leads identically to the least squares solution and the maximum likelihood solution. As such the estimates $\hat{\alpha}$ and $\hat{\beta}$ have the properties of

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consistency, efficiency, and sufficiency. Moreover they are unbiased.[1]

Now it is clear that the requisite sums and sums of products n, $\sum x$, $\sum x^2$, $\sum y$, $\sum xy$ in the above equations can be computed on a desk calculator. Also, it is well known that the accounting machine provides, with the use of progressive digiting,[2] an extremely fast method for computing these sums. A third method would be to use the 602, punching individual products as the calculation proceeds. I will not attempt to describe completely the conditions under which we choose one of the three methods described above. These general observations can be made, however: because of the increased opportunity for checking we generally prefer to use a punched card method even when the amount of data is small; second, since the calculation in the linear case is frequently only preliminary to later work, we are inclined to use either the 602 method or a combined 602 and 405 method. In this way we calculate individual products and save them in the card, and toward this end we have permanently wired 602 control panels for some of the curve fitting problems which we encounter frequently.

Several Fixed Variates

I will now discuss a more general case of curve fitting and in so doing I will indicate that in an important class of statistical problems it is necessary not only to compute the value of the inverse of a matrix but to compute explicitly the elements of the inverse matrix. We shall see, then, that the Type 602 methods of equation solving which Mr. Bell described are important in statistics and we will note also that it is convenient to augment the matrix of coefficients with the unit matrix in order that the elements of the inverse matrix may be obtained explicitly.

One example of curve fitting in several dimensions arises in the plastic industry. We denote by y the molecular weight of a plastic. We denote the operating conditions of a production process as follows: temperature, x_1; pressure, x_2; amount of agitation, x_3; time, x_4; amount of stirring, x_5; amount of monomer, x_6. We assume that $x_1, x_2, \ldots, x_6$ can be measured with perfect precision and that they can be fixed at prescribed values. We then perform an experiment in which n sample values $(y_j;\, x_{1j}, x_{2j}, \ldots, x_{6j})$, $j = 1, 2, \ldots, n > 6$, are obtained. If a linear relation of the form

$$y = a_0 + a_1 x_1 + \ldots + a_6 x_6$$

is postulated, then the problem becomes that of estimating the parameters $a_0, a_1, \ldots, a_6$.

Another example arises in the field of industrial medical statistics in which it is desired to predict the size of the dispensary staff and the amount of dispensary equipment needed in a new plant on the basis of knowledge of characteristics of employees; age, sex, race, occupation, education, salary, kind of chemicals worked with, etc. The medical aspects of this problem are under the direction of Dr. J. S. Felton, the mathematics under Dr. A. S. Householder, both of Oak Ridge National Laboratory, and the calculations include the inversion of a matrix of order fifty-five, these calculations now being performed on IBM equipment under J. P. Kelly and B. Carter at the K-25 Plant.

In general, we suppose that we have fixed variates $x_1, x_2, \ldots, x_k$, in which, for convenience of notation, we define x_1 to be identically equal to one. I will not formulate the problem in detail but rather will refer you to an excellent discussion of the problem by Wilks.[3] We make certain assumptions about normality of distribution of repeated measurements of y corresponding to fixed values of the x's, and we assume that there is a relation of the form

$$y = a_1 x_1 + a_2 x_2 + \ldots + a_k x_k \,.$$

On the basis of a sample of size n

$$(y_j;\, x_{1j}, x_{2j}, \ldots, x_{kj}),\ j = 1, 2, \ldots, n > k;$$

the maximum likelihood estimates and likewise the least squares estimates $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_k$ of the parameters are then given as solutions of the simultaneous equations

$$\begin{array}{l}
a_1 \sum_{j=1}^{n} x_{1j}^2 + a_2 \sum_{j=1}^{n} x_{1j} x_{2j} + \ldots + a_k \sum_{j=1}^{n} x_{1j} x_{kj} = \sum_{j=1}^{n} x_{1j} y_j \\
a_1 \sum_{j=1}^{n} x_{2j} x_{1j} + a_2 \sum_{j=1}^{n} x_{2j}^2 + \ldots + a_k \sum_{j=1}^{n} x_{2j} x_{kj} = \sum_{j=1}^{n} x_{2j} y_j \\
\ldots \qquad \ldots \qquad \ldots \qquad \ldots \\
\ldots \qquad \ldots \qquad \ldots \qquad \ldots \\
a_1 \sum_{j=1}^{n} x_{kj} x_{1j} + a_2 \sum_{j=1}^{n} x_{kj} x_{2j} + \ldots + a_k \sum_{j=1}^{n} x_{kj}^2 = \sum_{j=1}^{n} x_{kj} y_j \,.
\end{array}$$

Computationally, the first problem is that of computing the $k(k + 3)/2$ sums of products which are exhibited above. Second, we wish to solve the equations themselves.

We thus arrive at estimates for the parameters which are optimum in a certain sense but these estimates are single estimates or point estimates. If pressed for a single estimate we would give $\hat{a}_1$ as an estimate of a_1, etc. But statisticians prefer to give also a range of values for a_1, a range of values for a_2, etc., and to associate this range of values with a probability statement. For example, we might choose a confidence level of 95 per cent and then arrive at statements such as:

We assert with 95 per cent confidence that

$$\alpha_1 \text{ lies in the interval } (\alpha_1^0, \alpha_1^1)$$

$$\alpha_2 \text{ lies in the interval } (\alpha_2^0, \alpha_2^1)$$

$$\cdots\cdots\cdots\cdots$$

$$\alpha_k \text{ lies in the interval } (\alpha_k^0, \alpha_k^1).$$

I do not intend to describe either the theory of confidence intervals as developed by Neyman, Pearson and others or the theory of fiducial probability as developed by Fisher. Rather, I will say only that, in order to arrive at statements of this kind, it is essential to compute explicitly the elements of the inverse of the matrix of the coefficients of the equations exhibited above. In concluding this section, let me say that all of us have rule of thumb methods for deciding when to use hand methods and when to use IBM methods for solving linear equations. These rules must be modified in curve fitting problems because we must not only solve the equations but also compute the coefficients of the equations and we must compute explicitly the elements of the inverse matrix. Consequently, IBM methods are efficient for a lower value of k (number of equations) in the statistical problem than in the straightforward linear equation solving problem.

More General Forms

It should be mentioned that the mathematical models assumed in the previous sections can be thought of as including many, but not all, of the situations which one encounters in science and industry. For example, we note that parabolic forms can be obtained from

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k$$

by setting

$$x_1 = 1, x_2 = x_2, x_3 = x_2^2, \ldots, x_k = x_2^{k-1}.$$

However, there is another type of curve fitting problem whose solution requires a more complicated procedure than that given above but for which accounting machine equipment can still be used advantageously. I refer to a problem such as that of fitting an equation of the form

$$y = \alpha_1 e^{\beta_1 x} + \alpha_2 e^{\beta_2 x} + \ldots + \alpha_k e^{\beta_k x}.$$

In this case the equations to be solved in order to minimize the sum of the squares of the deviations of the observed from the calculated are not linear. Hence the method of successive approximation described, for example, by Deming[4] must be applied. But by this method we arrive at a set of linear equations whose solution leads to the numbers by which to adjust our first approximations. Consequently, what has been said above concerning the use of accounting machine equipment applies also to the present case.

Approximating Solutions to Partial Differential Equations

I shall only describe a problem on which we are working for Dr. A. S. Householder and Dr. B. Spinrad at the Oak Ridge National Laboratory. I should then like to ask for comments arising from your experience with problems of this sort.

We have the Laplacian partial differential equation in two dimensions with boundary conditions given on two squares, one square being located within the other. Because of certain symmetry conditions we need to consider the problem over only a triangular region. We then construct a lattice and replace the differential equation by a difference equation which relates the value of the solution at one point to the value of the solution at each of four neighboring points. By punching cards on which are designated the coordinates of the lattice points and the boundary values we then proceed in a combination of the operations of collating, reproducing, and calculating. The process is that of Gauss-Seidel and we find that, for one thousand points, we can perform an iteration in about two hours and a half.

Now the question which I have is whether any of you have a criterion as to when sufficient convergence has been obtained. Do you continue to iterate until there is no change in the final digit carried in your machine? Do you prescribe in advance an upper bound to the sum of squares of deviations from one iteration to the next? What is a good criterion?

In concluding this section I should say that many problems in industry can be attacked either by curve fitting methods or by analysis of variance methods. F. C. Uffelman and E. W. Bailey of the Y-12 Plant in Oak Ridge have developed efficient machine methods for applying analysis of variance techniques. I am sure that Mr. Bailey, who is at the Forum, would be glad to discuss that particular problem with you.

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DISCUSSION

Mr. Stanley: In regard to that last problem and the criterion required, I notice that the problem seems to be one which could be very easily adapted to the relaxation method of Southwell. I might mention a paper which gives a very good criterion in that case.[1] Offhand, I can't state whether the criterion can be easily adapted to the Gauss-Seidel method, but in view of the close similarity between the two methods it is quite likely that it can be.

In reference to Dr. Hurd's first problem: in that fitting of exponentials, I suppose by using least squares, the idea was to take a number of ordinates and, with the assumption that there were perhaps three terms in its series, to fit a least squares system to them. You arrive at a cubic equation and solve it for the three unknowns, whence the required values. The alphas are relatively easily derived.

However, I wonder how you attack that problem when the number of dimensions is very great—say twenty or thirty. Then least square fitting becomes very laborious.

Dr. Hurd: Your first comment, as I understand it, is about whether the Gauss-Seidel method is a relaxation method. They are usually talked of as being one and the same.

Dr. Thomas: I should like to say that while many people use the names indifferently, I think it is well to make the distinction that if you iterate regularly you are using the Gauss-Seidel method; if you always improve the worst point you are using the relaxation method.

Dr. Hurd: In reply to Mr. Stanley's second question, we wouldn't have ten ordinates. We would probably have a thousand or perhaps two thousand. Actually the least squares method is the way we would approach the problem.

Dr. Tukey: With regard to this problem of fitting the exponentials, there is an old paper in the Vienna Academy Proceedings,[2] when people were first starting to untangle radioactive chains, which I think probably isn't as well known as it should be. It amounts to this, that if you take your x values, as if you had taken a square network and projected it down in a slanting manner, then you can solve these in a particularly simple fashion.

Now where you have a thousand values, you ought to be able to sort out one hundred which approximately meet these conditions, and get a rather simple and accurate first set of values.

Mr. Stanley: As for this relaxation method, I wonder if you have had any experience in trying to adapt it to punched card equipment.

As Dr. Thomas has pointed out, the relaxation method is not well suited to this equipment, as it goes from one point of the net to another in what might be considered a haphazard manner. Possibly punched card machines can be used in a satisfactory manner in conjunction with a system of block or group relaxations.

Dr. Thomas: In regard to using relaxation methods, it is more advantageous, at least for convergence, to go over the network by hand, make the adjustments directly, and then compute the residuals on the machine. That is the main advantage of the relaxation method; adjusting the worst points gives more rapid convergence.

Mrs. Rhodes: Did you try the harmonization scheme? I have never tried it on the IBM equipment, but I have an idea with the Type 604 we might save a great deal of time using this scheme. Moskovitz[3] and Frocht[4] gave this scheme for just such areas as Dr. Hurd described; it yields a first approximation in no time at all. Emmons[5] gives a problem very similar to the one Dr. Hurd showed, as an illustration of the superiority of the relaxation method over the Liebmann transit method of solution. As I recall, the times for the respective solutions were given by him as 2½ and 11 hours.

Dr. Grosch: To do single point relaxations, one can use both feeds of a collator to search for the largest residual in a deck of cards. That means it is possible to survey eight residuals a second: faster than a human operator by almost a whole order of magnitude. A twenty by twenty mesh could be reviewed in two minutes. The catch is that the machine does not skip right to the troublesome areas of the mesh in the way the human eye does.

Mr. Stanley: It really seems that, because of the time consumed, a system of single point relaxation is out of the question. Group relaxation might do for a part of the problem.

Dr. Grosch: Unfortunately, group relaxations don't work out too well in terms of standard equipment. I am afraid it requires a sort of human thinking that the 604 and the collator won't do. The SSEC, perhaps, might.

Dr. Thomas: I would like to remark that just as in the problem of ordinary differential equations we talked about earlier, you can get a simple formula. The error term is in the sixth order and actually computing is no more complicated. The same thing is also true of partial differential equations in two or three or more dimensions; in two, if you take an improvement formula that involves only the four values at the corners of the square in addition, you can get a formula which is accurate to error terms in the sixth differential coefficients; it will enable you to get accurate results with a much coarser mesh and will save an enormous amount of time where you have a large number of points.

Mr. Bell: Getting the inverse of a sixty by sixty matrix is a big problem any way you look at it.

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Some Engineering Applications of IBM Equipment at the General Electric Company

FRANK J. MAGINNISS

General Electric Company

WHEN I was asked to give this talk, it was suggested that I describe one of the problems for which we had employed our IBM machines. Since we have two sets of these machines being used for engineering calculations on which some variety of problems has been worked out, I thought it might be of interest to give a more comprehensive picture of what use we are making of these machines, without, however, going into much detail for any one application.

Our two groups of machines are operated by the Turbine Engineering Division and by the Central Station Engineering Divisions. The former set is used for the calculation of shaft critical speeds and for the solution of a problem which will be described by Mr. Kraft in his talk. The other set has been used for a greater variety of problems. I shall try to describe these problems (with the exception of Mr. Kraft's) to indicate why we felt IBM machines would be useful to us at the General Electric Company.

Critical Speeds of Turbine Shafts

One of the important problems facing the designer of large turbine sets is that of determining the critical speeds of the set. Critical speeds are those rotative speeds which coincide with the natural frequencies of transverse oscillation of the shaft. A better set from a vibration standpoint is obtained if the critical speeds are not too close to the running speed.

In the past, these criticals were approximated by considering that each unit between bearings (the high pressure turbine, the low pressure turbine, and the generator) was a separate entity, and the lowest frequency critical speed of each of these single spans was calculated under this assumption. This method was used only because no other method was available which would give results in a practical length of time.

Recently, our Turbine Engineering Division decided that a more accurate determination of the critical speeds of the multiple-span shaft involving a large number of calculations could be made in a satisfactory length of time by using IBM machines. Accordingly, a set of these machines was installed, and for the past two or three years every new turbine set which has been built at the General Electric Company has been previously analyzed for the location of the critical speeds of the shaft. Moreover, all sets which have been built in the past and are now in service are being analyzed. In all cases in which test results are available, they agree very closely with the results of calculation.

Briefly the method of calculation is this: the shaft (and by shaft is meant the shaft and the units on it) is assumed to be made up of a definite number (about thirty) of concentrated masses. Equilibrium equations are written for each of the masses for the forces and moments on it due to the reaction of adjacent masses and the centrifugal force due to rotation about the unstressed axis of the set. A speed of rotation is assumed, and starting at one end the deflection, slope, shear, and moment are calculated for succeeding sections until the far end of the shaft is reached. Unless the speed assumed is exactly a critical speed, it will not be possible to have both the shear and moment zero at this point. If the shear is made zero, there will be a residual amount which may be positive or negative. When a number of such calculations as described above has been made for a number of speeds of rotation (at about 200 rpm intervals for a 3600 rpm machine) the curve of residual moment vs. shaft speed will indicate by its zeros the desired critical speeds.

Correlation of Data on Electrical Steel

Obviously a factor of great importance to the manufacturer of electrical equipment is the magnetic characteristics of the steel he receives from various fabricators. Any significant departure from a standard may have serious consequences in its effect upon the performance of a rotating machine or a transformer. It is therefore a question of some interest to the people in our laboratories what factors are responsible for such deviations from standard. Are they caused by small variations in the amounts of carbon, manganese, cobalt, silicon or other elements which enter into the composition of the steel? Or are they caused by slight differences in the heat treatment the steel may be given by different manufacturers or by the same manufacturer at different times?

An attempt was made to answer the first of these questions by determining the correlation between the percentage of each of the elements present in the composition of the steel with certain magnetic properties and by making a multiple correlation between overall composition and the magnetic properties. This particular study showed only a small correlation between chemical composition and magnetic properties.

This problem afforded an excellent opportunity to use the technique of progressive digiting on the accounting machine.

Multiple Conductor Circuits

A recent trend in the field of transmission of electric power has been in the direction of higher voltages. A reason for this is the increased distance between power source and load center. Tests up to 500 kv are being conducted to determine the feasibility of higher voltage transmission.

When high voltages are to be used, there appear to be certain advantages in using more than one conductor per phase. For example, the line inductive reactance will be lowered as multiple conductors per phase are used and the capacity susceptance increased. These result in an increased load-carrying capacity of the line for the same voltage. It is of interest to determine the spacing and arrangement of conductors in one phase and between phases which will prevent excessive corona loss. In order to determine these it is necessary to know the maximum value of the voltage gradient which will occur at any of the conductors. This in turn depends on the instantaneous value of the charge on the conductors.

The problem is to set up and solve the equations relating the charges appearing on each of several conductors with the voltages on the conductors. In the general case there will be as many such algebraic equations as there are conductors. Moreover, in order to determine the maximum gradient it will be necessary to know the charge distribution for several values of the voltages. For a two conductor per phase, three phase line with two ground wires, we are faced with eight algebraic equations. Each variation of conductor size or spacing which is to be investigated will require a new set of equations with different coefficients. The coefficients are Maxwell's potential coefficients and the equations may be readily set up. It then becomes a question of matrix inversion for the reason mentioned above. An interesting and significant result of the study was the fact that the maximum voltage gradient can actually be decreased for a two conductor per phase line over that for a single conductor of the same total area per phase.

Incompressible Fluid Flow

For many field problems it is possible to obtain a very close approximation to the stream lines and equipotential lines by using some sort of network calculator such as the AC Network Analyzer or the DC Calculating Board. As an example, Concordia and Carter[1] determined the fluid-flow pattern in a centrifugal impeller by this method. However, most network devices are inherently limited in range and accuracy (as are all simulator type calculators). If, therefore, it is required to determine flow lines to a high degree of accuracy in a region of great curvature, a digital method of solution is necessary.

The reported results of the problem of the centrifugal impeller study just mentioned were further refined, using IBM machines to improve their accuracy. A small region around the trailing edge of the blade was subdivided into a fine mesh of points, values of the stream function of the boundaries of this region were assumed to be fixed at the values obtained from the DC Board study, and values of the stream function at interior points of the region were improved by a relaxation method.

Incidentally, we believe that the only way to study a fine mesh relaxation problem is to start out by setting it up first on the AC or DC Network Analyzers using as many units as possible to give reasonably close starting values. If such a device is unavailable, it would be necessary to start with a very coarse network and gradually increase the number of points until the required fineness of the mesh is obtained.

In addition to the work done on the centrifugal impeller, a study is at present being carried out to determine the flow lines around a turbine bucket. In principle the two problems are identical.

Some other problems we have been or are at present studying by means of our IBM machines are:

1. A study of the distribution of flux density in the interior of a steel lamination under conditions of iron saturation. Such a study could be valuable in predicting transformer losses to be expected from a new steel. This problem involved solving one of Maxwell's equations in the steel subject to the condition that the total flux in the steel be sinusoidal in time. This is a rather tricky condition and difficult to satisfy since the best we could do was to assume the flux density at a given point, for example at the center of the lamination, as a function of time and by

means of a difference equation determine the flux density at successive points from the center out to the edge. If the total flux was not sinusoidal in time, a different central flux density was assumed and the procedure repeated. Actually, of course, we did a number of cases at the same time. This problem also led us to develop a method for harmonic analysis for odd harmonics only.

2. A study to determine crystal structure is being carried on at the present time. The method we are using has been described by Shaffer, Schomaker and Pauling.[2]

3. Finally, we endeavored to solve a small but complicated circuit problem involving complex voltages, currents and impedances to which several different frequencies and values of load were applied. Our conclusion from this study was that the ratio of the number of steps involved in the solution to the number of cases considered was so large as to make this particular problem unsuited to IBM machine solution.

I think I may state our conclusions as to the use of these machines as follows. Our set of machines on which the critical speeds are being calculated is being used very efficiently. Although there are not many cases (that is, values of frequency) for each turbine set studied, the successive steps are identical, control panels are permanently set up, the operators are thoroughly familiar with the procedure, and the work grinds out day after day.

The other set of machines on which the remaining problems I have mentioned have been worked out cannot, obviously, be used so efficiently, since we must be prepared to solve many different kinds of problems. Of the types of work we have carried out we believe the most fruitful use of the machines compared to other means of performing the same calculation has been in the field of solving algebraic equations or matrix inversion.

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DISCUSSION

Mr. Stevenson: On the turbine blade problem, we have worked a somewhat similar problem using basically the method of Theodorsen and Garrick[1] for arbitrary airfoils. By means of a prepared table, we are able to get pressure distribution on an arbitrary airfoil in approximately fifteen minutes' work with a 405 only. That method can be further expanded to give you either the velocity potential or any streamline you wanted to specify. I would be glad to let you know about the method as I think it would reduce your machine time.

Dr. Caldwell: I would like to mention two points. Mr. Maginnis briefly touched on a subject which I believe has considerable possibility in many of these jobs, and that is the possibility of using a DC board to get a start on a numerical solution. Likewise, one can use electronic analog machines of low precision but high speed to get a rough approximation, and then jump off from there. I think that is a very powerful type of combination. If any of you are doing this, and making it work, I would be interested in knowing it.

Dr. Fenn: In some work which I may describe later at these sessions, precisely that was done. We have converted a gun director into an analog machine, and sets of suitable approximate solutions for certain iterative digital processes are actually worked out in large numbers. It is, as you say, a very powerful method and saves a tremendous amount of time.

Dr. Caldwell: Another thing I would like to see somebody do some day is to work some fundamental improvement on the method of crystal structure analysis that has been described. It isn't too happy, the way you have first to draw out a lot of cards—picking them by hand is about the only practical scheme—and run around the place juggling little groups of cards. Unless you have a very conscientious clerical job done, you are going to get something mixed up. At least I haven't seen it work yet without depending a great deal on the human element to keep things straight.

Dr. Thomas: I would like to say we have been developing much smaller files of cards for these various harmonic analysis calculations, and we are proposing to do it entirely by sorting, collating, gang punching and tabulating. It can be done that way, and in many cases is quite as fast as using many cards. If you have a really fast multiplier, such as the 604, you could probably do it quicker with that, but with only the 601 or even the 602 available, it is quicker to do the multiplying by progressive digiting.

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Planning Engineering Calculations for IBM Equipment

BEN FERBER

Consolidated Vultee Aircraft Corporation

CONSOLIDATED VULTEE started back in 1942 to use standard IBM equipment to help solve engineering problems. The first use was on electrical load analysis. The solution of electrical load problems continued until about 1944, when the use was extended to structural problems. Because of the saving in time and money, the equipment has been in use continuously since that time, and has been applied to a wide variety of engineering problems.

We have found that it is a good idea to get an estimate from the person who wants a job done as to how long it would take to do it manually. Then we put it on IBM machines if we are sure that we can at least better his cost estimate. Another important point to consider is the advantage of having the engineer cooperate and assist in getting the job on the equipment. We must replan his job to suit the equipment. In fact, with his assistance we might discover new techniques and provide him with a better service than he requested or anticipated.

For a particular problem the first two columns of the card are used to identify the problem. The card layout form with the card code is filed for possible later use on a new problem of the same type. The card code was very helpful to us in looking back to find the cards that represented a particular job. It could also be used to sort cards into sequence, or for control on various machines both as to its digit and as to its zone punch.

After the war we had a large supply of control panels, so we used a large number of permanent control panels. When the time came that we ran short of permanent panels, we have gone to semi-permanent panels. For a 602 where we have a great deal of basic wiring that does not change, we use a few manual wires. Perhaps some of you have considered such a possibility for a 405 differencing control panel, using manual wires to make minor changes.

Another use of manual wires combined with permanent wires might be in wiring a panel for the first time. Testing the panel after a few steps and substituting permanent wires for the manual wires enables us to see what we are doing.

Many of our jobs use relatively small quantities of cards but many steps. For these jobs it is often a time saver to wire a few simple control panels rather than a single one that is very complicated. Of course, we can afford to spend more time on the wiring of a complicated board if it is to be a repetitive job, or if it is very large.

We have very good cooperation with our accounting machine installation. Having our installations together does have certain advantages. We can have our multiplying done on one machine and checked on another.

Although the actual procedure used to solve a given problem would to a large extent be a function of the type of machine that is available, the following approach to a common structural problem may prove interesting to some of you. Let

$$f_b = -\frac{(M_y I_x - M_x K)\,x}{I_x I_y - K^2} - \frac{(M_x I_y - M_y K)\,y}{I_x I_y - K^2}$$

where

f_b = stress at any point whose coordinates are x and y, measured from any pair of rectangular axes passing through the centroid of the cross section

M_x and M_y = bending moment about the x-axis and y-axis, respectively,

I_x and I_y = moments of inertia of the cross-section about the x and y axes

K = product of inertia of the cross-section about the x and y axes.

If

$$C_1 = \frac{1}{I_x I_y - K^2}$$
$$C_2 = C_1 K$$
$$C_3 = C_1 I_x$$
$$C_4 = C_1 I_y$$
$$D_x = x\,C_2 - y\,C_4$$
$$D_y = y\,C_2 - x\,C_3$$

then

$$f_b = M_x D_x + M_y D_y\ .$$

What we did was simply to put x and y inside the parentheses and take out the bending moments. Most of the time we wish to hold the section constant and solve stress for various combinations of bending moments. If we wish to continue and solve for shear flow, the basic formula for shear flow q due to vertical and side shear in an open section is

$$q = \frac{(V_x K - V_y I_x)\, Q_y}{I_x I_y - K^2} + \frac{(V_y K - V_x I_y)\, Q_x}{I_x I_y - K^2}$$

where

V_x and V_y = shear forces perpendicular to the x and y axes

Q_x and Q_y = static moments about the x and y axes

$$Q_x = \sum_i A_i y_i$$

$$Q_y = \sum_i A_i x_i$$

A_i = area of the cross-section of the ith item

$$\sum_i D_{ix} A_i = \frac{K\, Q_y}{C_1} - \frac{I_y\, Q_x}{C_1}$$

$$\sum_i D_{iy} A_i = \frac{K\, Q_x}{C_1} - \frac{I_x\, Q_y}{C_1}\ .$$

Then

$$q = V_x \sum_i D_{ix} A_i + V_y \sum_i D_{iy} A_i\ .$$

DISCUSSION

Dr. Caldwell: I especially appreciate Mr. Ferber's idea of getting the customer to estimate how long it would take to do the job. It is very common to find a customer arriving at one of these installations that are supposed to do mass production computations with his eyes bigger than his stomach. He says, "Well, we can get a lot of work done here for practically nothing," so he specifies a lot of work. Later he says, "Why do you say this is going to cost thousands of dollars? I could do it for a hundred dollars." You investigate a little further and find he is going to do one per cent of the work for one hundred dollars!

Mr. Ferber: I might add that when a problem is worked on standard IBM equipment with no extra devices other than those normally available, one operator does about twelve machine-hours of work per day.

Mr. Bell: The indirect saving resulting from the elimination of rework in production can be a very important result of using IBM equipment for engineering design. This follows directly from the ability to investigate many design conditions quickly and completely, so that all design parameters are well in hand before construction begins.

A Survey of the IBM Project at Beech Aircraft Corporation

JOHN KINTAS

Beech Aircraft Corporation

THE PRESENT PAPER is intended to describe some important problems being solved by the IBM installation in the engineering department of the Beech Aircraft Corporation. Emphasis is placed on the various types of jobs processed by the IBM group for our engineering and sales departments. Consistent with the stated objective of the Forum, particular emphasis is placed on problems which arise frequently in structural engineering. It is hoped that the ideas contained herein will help stimulate discussion and thereby foster a mutual exchange of ideas.

Introduction

At the present time the computing installation in Beech's Engineering Department includes one each of the following International Business Machines:

Type 601 Multiplier
Type 513 Reproducing Punch
Type 080 Sorter
Type 405 Accounting Machine
Type 552 Interpreter
Type 031 Alphabetic Key Punch

The need for such an installation was envisioned during the war as a practical solution to some problems arising from critical manpower shortages. It was considered that the IBM group would function in the following ways:

1. Accommodate certain types of periodic and intermittent work-loads of our engineering department.
2. Alleviate shortages and losses to the armed forces of skilled technical personnel.
3. Help relieve engineering personnel of routine calculations, thereby providing time for more important duties.

To date our IBM group has handled a considerable amount of work relating to airplane weight control, structural analyses, sales research, time-labor studies, field service engineering, library records, and other engineering problems. Some of these projects are discussed herein. Consistent with the stated area of discussion, however, particular emphasis will be placed on technical problems of aircraft engineering.

Weight Control Problems

Of importance is the current usage of machine methods in weight control calculations. In much of this type of work the machines are required to shoulder intermittent, heavy work-loads, and have demonstrated appreciable savings in time and labor. In a general sense, our weight control group determines the conditions of weight and balance of the various Beech airplanes. In particular, this group investigates the weights and center of gravity locations of composite airplanes for various loading conditions. Such information usually is obtained by considering the weights and centroid locations of all fixed and movable items of mass in the airplane. Punched card methods are readily adapted to handle the large volumes of weight and balance calculations associated with this type of work.

Sales Research Problems

Our computing installation was used recently in a sales research program to determine potential markets for Beech products. This program was essentially a statistical survey of numerous parameters relating to sales potentials in various geographical areas. The labor entailed in the mathematical manipulation of these parameters was considerable. However, many basic operations were repeated, and therefore adapted to punched card methods of solution. The machines relieved the sales research group of many hours of "donkey work."

Structural Problems

We are set up to handle the following structural engineering problems by machine methods:

Three-dimensional flutter analyses of aircraft structures.

Harmonic analysis.

Solution of linear simultaneous equations.

Solution of complex matrix equations.

Computation of section properties of aircraft structures.

For each of these problems we have provided our IBM group with a master deck of precoded cards, wiring diagrams, and a set of written instructions. The master decks are considered as permanent equipment, that is, they are not processed in the solution of a particular problem. For specific problems, the master decks are reproduced to obtain working decks of cards which are processed in accordance with the written instructions. The instruction sheets we use at Beech avoid, wherever possible, reference to technical significances of the steps being performed. This divorcing of engineering aspects of a problem from the required machine operations enables the IBM operator to concentrate mainly on manipulation of the cards.

Flutter

The punched card method of flutter analysis we use is based on the theory given in ACTR No. 4798.[1] Standard procedures have been set up for the following basic flutter modes:

1. Fixed surface bending vs. fixed surface torsion vs. rotation of control surface (with or without geared tab).
2. Fixed surface bending vs. fixed surface torsion vs. airplane roll or vertical translation.
3. Fixed surface bending vs. fixed surface torsion vs. rotation of control surface (with or without geared tab) vs. airplane roll or vertical translation.

These flutter modes are solved entirely by the machines, except that a few manual operations are required during the final stages of solution.

Usually, flutter analyses are conducted to determine the critical flutter speeds and frequencies and the associated values of damping coefficients. In some cases, additional information such as mode shapes and amplitude ratios of the component degrees of freedom also may be required. The problem of determining these items may be resolved into the two rather distinct phases of formulating and solving the stability determinant. We formulate the determinant by straightforward operations on matrices, then solve the determinant by trial-and-error iteration. Some discussion on these important steps is considered desirable.

For any mode of flutter, the stability determinant is composed of complex elements. The numerical value of each element may be determined by evaluating and summing up certain aerodynamical and mechanical integrals. Careful study has shown that:

1. Each of these integrals may be expressed as a summation of products of finite quantities. This would be equivalent to considering that the wing is divided into a finite number of chordwise strips.
2. Multiplication and summation of these products may be accomplished readily using methods of matrix algebra.
3. The numerical value of each element of any stability determinant may be expressed as the product of four matrices.

On the basis of these observations, we form stability determinants entirely by machine methods.

As previously mentioned, we solve the flutter determinant by trial-and-error iteration. It has been determined that this process will converge at a practical rate, since the preponderant elements usually lie on the principal diagonal. In most cases the iteration stabilizes satisfactorily in two or three trials. However, in some cases four or more trials may be required. In general the iteration process is carried out in the following way:

1. First a trial value of ω is substituted in all but one of the elements along the principal diagonal.
2. The determinant is reduced to the third order, if necessary, then expanded to obtain a linear equation in one unknown.
3. The linear equation is solved to obtain the second trial value of ω.
4. Steps 1, 2, and 3 are repeated until the trial value of ω agrees with the solution.
5. The entire process must be repeated a number of times equal to the number of degrees of freedom considered.

Practically all of the labor of solving the stability determinant is done by the machines. However, some manual operations are required to estimate initial trial values of ω and calculate flutter speeds, frequencies and damping coefficients at the final stage of solution.

Estimates of the time required for the solution of a three-degrees-of-freedom flutter mode by manual methods as compared with our punched card method, based on the assumption that five values of $(v/b\omega)$ are investigated for each mode, are as follows:

IBM Time	*Operation*	*Manual Time*
35 hours	Computation of elements of the stability determinant	30 hours
26 hours	Solution of the stability determinant	44 hours
61 hours/mode (1 operator)	Total time	74 hours/mode (1 person)

Waveform Analysis

Complex periodic waveforms frequently occur on vibration records of structural investigations such as flight testing, fatigue testing and vibration tests of power plant installations. It is impossible to analyze many of these waveforms by ordinary inspection methods. However, any complex periodic wave may be represented by the superposition of a number of simple sine and cosine waves. It might also be mentioned that aperiodic curves, continuous in a finite interval, also can be represented by assuming that the given curve represents a single cycle of variation.

The problem of waveform analysis is primarily that of determining the amplitudes and frequencies of the sine and cosine components present in the synthesized wave. At any point along the reference axis, the ordinate of a composite waveform is equal to the sum of the ordinates of the component harmonics.

At Beech, we have expanded the Fourier series to obtain the general equations corresponding to five, eleven, twenty-three and forty-seven harmonics. Particular solutions may be obtained by substituting into these expansions the numerical values of the ordinates of a given curve.

We have transferred the trigonometric constants of these expansions into a master deck of 4704 coded cards. Solution of a specific problem may be obtained by punching into a working deck (reproduction of the master deck) the measured values of the ordinates. The cards are then processed in accordance with standard instructions. Solution is accomplished almost entirely by the machines; some divisions and extractions of square roots must be performed manually.

A comparison of time required by manual and punched card techniques for waveform analysis is of interest.

Number of Ordinates to Curve	*Corresponding Number of Harmonics*	*Est. Manual Time-Hours*	*Est. Machine Time-Hours*
12	5	2.0	1.0
24	11	7.5	2.5
48	23	28.0	7.5
96	47	110.0	26.0

Linear Simultaneous Equations

Our IBM group now is set up to solve systems of fifty linear equations in fifty unknowns. While this number accommodates our present needs, it can readily be expanded to any practical limit.

In general, we utilize a modified Gauss method. Here, the linear equations are converted into matrix form. The matrix equation, on the left side, contains a square matrix of constant coefficients postmultiplied by a column matrix of the unknowns. The right-hand side of the equation has only one column matrix of constants. The square matrix of coefficients is operated on by rows and by columns until all terms below the principal diagonal are zeros and each term along the diagonal is unity. During operations on rows, the column matrix on the right side of the equation is also modified. It is clear that the value of the nth unknown is immediately given on completion of these operations. The value of the nth unknown then is employed in a back-tracking process to determine the $(n-1)$th unknown, and so on.

It has been determined that time saved by machine methods over manual methods increases appreciably with increasing numbers of equations and unknowns. This may be seen from the following comparison:

Number of Equations and Unknowns	*Approximate Time for Solution—Man Hours* IBM by Decimals	IBM by Powers	*Manual*
10	Approximately Same		8.5
30	39	50	74
50	78	148	218

Matrix Manipulation

We are equipped to handle certain types of matrix equations by the Kimball method.[2] It is particularly useful in performing the basic operations of matrix algebra on matrices with complex elements. The method is also useful in manipulating matrices with variable elements when the solutions are approximately known or when the preponderant elements lie along the principal diagonal. This latter feature is a reasonable guarantee that trial-and-error iteration will converge at a practical rate.

Other Projects

At the present time we are investigating the feasibility of solving the following structural problems by punched card methods:

1. Spanwise airload distribution for monoplane wings.
2. Natural torsional frequencies of crank-systems using Holzer's technique.
3. Natural uncoupled frequencies of beams using Stodola's iteration procedure.
4. Analysis of shear lag in aircraft structures.

Time Aspects

The time estimates previously given for several structural problems were based on actual performances. They were determined by solving given problems manually and by machine. Now the approximate average rates of our machines are as follows:

Multiplier—15 cards per minute
Reproducer—100 cards per minute
Sorter—450 cards per minute
Accounting Machine— 80 cards per minute (detail print)
150 cards per minute (group print)
Interpreter—60 cards per minute

Processing time for most problems can definitely be improved through usage of faster calculating punches. We probably will consider faster machines when the need for more rapid processing becomes manifest.

Recommendations

Usually, punched card procedures can be adapted to a given engineering problem in a number of ways. From time and labor standpoints some procedures will be more efficient than others. Among other factors, a determination of the optimum procedure depends on a knowledge of the full capabilities of the machines available for our use. For this knowledge we rely to an appreciable extent on the Wichita staff of International Business Machines Corporation. We have always found them highly cooperative. However, in some cases they were unable to provide us with enough information on specialized capabilities of the machines, particularly our Type 601 Multiplier.

We recommend that local IBM offices be provided with up-to-date information on the full computing capabilities of the machines in their region. It may be possible to establish, on a current basis, the flow of such information from the various IBM research laboratories and computing centers to branch offices. This would help people like ourselves to realize the maximum potential utility of IBM installations and avoid needless duplication of effort.

References

1. B. Smilg and L. S. Wasserman, "Application of Three-Dimensional Flutter Theory to Aircraft Structures," Materiel Division, Air Corps, *ACTR* 4798 (1942).
2. E. Kimball, Jr., "A Fundamental Punched Card Method for Technical Computations," Bureau of the Census, U. S. Department of Commerce (undated).

DISCUSSION

Dr. Eckert: I might point out that the use of IBM machines for technical computation grew very slowly for a number of years. It was very easy for two or three people to keep in touch with each other. Within the last two or three years, there has been such a sudden cloudburst that to have people everywhere supplied at the right time with the right information is a little difficult. Steps are being taken; IBM has now, among other things, special representatives in the Sales Department who understand what you are trying to accomplish. They know what is available in IBM, and they are at your call.

With respect to the local manager, he also has a very tough assignment when you ask him for methods of doing things he has never heard of. That gap has to be bridged, and I am sure it will be in a very short time.

There is still another way which is open at the moment, and that is to call on us at the Watson Laboratory or write us a letter. We are always glad to hear from you. I think about half of the people in this room have already done that.

Mr. Kintas: You probably know Mr. Kimball of the IBM office in Dallas. Would it be possible for him, or others like him, to circulate periodically as good-will ambassadors? A person like that could not only convey information from your laboratory to our installations, but could pick up ideas from us to be passed on. Such representatives would be very valuable.

Dr. Eckert: That is the intention. The difficulty is to get the right man, get him trained in a very difficult field, and get him to you.

Dr. Korn: Why not send Dr. Grosch?

Mr. Schroedel: Mr. Kimball, Dr. Grosch, and I participated in a rather successful experiment along those lines recently. Lectures were given at the Cornell Aeronautical Laboratory in Buffalo, and people from other computing groups in the vicinity also attended. We learned a good deal from the meeting. But there are fifty or sixty installations in various parts of the country, and it is not easy for all of us to visit every locality and work with you long enough to really make a contribution. We hope to have more technically trained representatives in the Sales Department as time goes on.

Mr. Bisch: Several points, which could stand some comment and emphasis, were picked out of the very interesting talks of our aircraft representatives, Messrs. Ferber, Bell and Kintas. Instead of taking one point at a time, it seems more constructive and brief to present all my comments as a whole.

Many engineering organizations are looking today for powerful means of calculation. What the Engineering Department of North American did four years ago was to look to IBM for the main answer to this need, after a general survey of the field. Fast progress was desired, and to this end production methods were used in the creation and the development of our Engineering Section of accounting machines.

Such methods call mainly for extensive specialization and perfect coordination. To be concrete, we select the structures section, although the following would be true for the aerodynamics section as well.

An engineer with a long acquaintance with problems of structures, company policy, and organization methods was asked to select the problems and among their various solutions those which result in speed and efficient use of the IBM machines. This engineer familiarized himself with the various functions of the IBM equipment and the pattern of calculation most suitable for it, but he made no attempt to learn how to operate it. As a result, he was able to increase the contribution of the machines and he promptly reached the conclusion that for a given standard problem, the machines can do everything from the punching of the initial data to the final report printing. It is important to remark that he was the only engineer in direct contact with the accounting machine section.

On the other hand, an IBM operator with several years of experience and a mathematical knowledge equivalent to a master's degree was assigned to program for the machines the problems offered by the structural engineer, to propose profitable changes in the mathematical processes, to suggest further use of the equipment, and to select the type and the number of IBM machines. In no case did he concern himself with the engineering aspects of the problems.

When this work was sufficiently under way to call for more than two operators, an experienced operator was selected as supervisor of this group. This group is an independent section of the main accounting section with its own machines, and it derives many obvious advantages such as readily available service for the machines and the incidental facilities of a large installation, by not being separate from the main IBM body. The two men at its head, the mathematician and the supervisor, form a perfect team for the dual purposes of large volume and continuous improvement.

The Engineering Section is therefore made of two parts: the engineering part and the accounting machine part, through which a perfect coordination is possible by a one-man contact. It is now opportune to detail further the duties and achievements of each component.

The engineering part, which is also assigned to seek new technical and experimental solutions of aircraft engineering problems, to write reports on the algebraic and tabular solutions of problems and to conduct experimental work, keeps an accounting of the approximate number of arithmetical operations required by every job sent to the IBM section. This number is easily arrived at for standard problems, using simple algebraic formulas. On the other hand, an extensive survey has shown that an engineer can perform an average of one hundred arithmetical operations for each remunerated hour. As shown by two years of coverage, an IBM operator performs on the average of one thousand such operations per hour; therefore his speed is ten times greater.

Finally, last year's operations show an average of seven thousand engineer-hours per month performed by an IBM Section of four operators, which would cost a minimum of twenty-one thousand dollars per month, if performed by engineers, against an over-all cost of six thousand dollars by the IBM group, thus effecting a saving of fifteen thousand dollars per month. This approximate saving of seventy-five per cent which was evidenced at the very beginning was the real selling point to our management.

However, the engineering management has become conscious of other less tangible although equally important advantages. They are:

Unprecedented dependability from the point of view of time and accuracy.

An average speed ratio of ten resulting also in cutting down waste in fabrication, as pointed out by Mr. Bell.

Availability of solutions of problems, previously prohibitive on account of cost and time.

I should like to conclude by giving a list of several questions frequently asked, and brief answers.

What is the accuracy generally used? Eight, sometimes ten, significant figures.

Why would slide rule accuracy not be a factor in this selection? Because in extensive calculations, even when performed in tabular form and on desk machines, there is a need for mathematical and engineering checks, which would be meaningless with slide rule accuracy. In some problems, such as the solution of systems of linear equations, more accuracy sometimes means the only chance of getting a correct answer.

Why not ask every engineer the estimate of engineering time, as practiced by Mr. Ferber? Because it is easier to gather in one room all the supervisors and experienced stress men and have them propose once and for all a sober rounded-off average.

Why not teach engineers how to operate the machines? Although they are all welcome to visit the installations and to see the machines performing, they only are interested in the relief in their task provided by use of the equipment. Maximum efficiency is obtained by showing them concrete types of calculations suitable or unsuitable to these machines; even this is not necessary for the standard problems. Finally, all the past experience being available to the contact engineer, it can be poured into every new problem.

What is the criterion for a solution by the machines? The IBM section can solve a mathematical problem in any way selected for efficiency, provided the final answers would coincide with the answer found by engineering methods.

Is the most efficient method reached the first time? Far from it. First of all, any substantial increase of efficiency is worthwhile at any time because the gain will be repeated many times and will outweigh the cost of the changes. Moreover, the survey of the first numerical application of a method always suggests other improvements.

Wouldn't the IBM section soon be confused by a maze of various methods? On the contrary; thanks to the specialization of the engineering group, it has been possible to restrict them to a few general solutions from which many problems can be derived as simple cases without change of procedure. For instance, the bending and shear stress distribution of shell structures can be obtained by the same process, regardless of the type of cross-section and the part of the airplane. Several obvious advantages are attached to such generalization.

What types of solutions are preferred, the type which is short but inefficient on the machines, or the long but efficient type? The first type is generally adapted to desk machines. Our experience shows that the last type is preferable, because it taxes the operator less, and usually the answer is reached sooner.

What aeronautical problems were found to be solved efficiently?

1. Conventional and elastic determinations of stresses and rigidity of shells.
2. Weight balance, static and dynamic.
3. Airplane performances.
4. Determination of design loads, magnitude and distributions.
5. General problem of arch analysis, stresses and deflections.
6. Redundant structure analysis.

What mathematical problems?

1. Solution, by iteration methods, of characteristic equations such as in vibration problems.
2. Direct solution of systems of linear equations up to seventieth order.
3. Solution, by Galerkin's method, of systems of linear differential equations with variable coefficients.

What is our trend with respect to wiring control panels? It may be that the nature of our work calls for flexibility. In any event, more efficiency is attained by breaking the process of solution into simple steps which are made with standard control panels, than with special panels aimed at shortening such processes.

Are we trying to use a minimum of cards? In general, one card is used for one machine operation. The cost and inconvenience of more cards is outweighed by the advantages of simpler procedures.

What is the future of such machines in aeronautical engineering? As long as this industry secures experimental contracts an increasing need for their use will exist.

Has use of IBM machines contributed to the progress of aeronautical engineering? Very much. Experimentally, thousands of stresses are daily recorded directly in cards from a specially wired stress rosette recorder. Technically, modern airplane structures can now be correctly analyzed at reasonable cost, whereas before the advent of the machines, this would have been impossible. Moreover, the mentioned recorded stresses can be rapidly transformed, through matrix calculations, into information valuable for future designs.

Could the progress in use of IBM equipment for aeronautical engineering be readily used for other engineering specialties? For problems with identical mathematical equations the answer is obviously yes. As to engineering problems such as all general structure calculations (like shell and arch analysis), their IBM solution can readily be used in civil engineering, when presented under the general form which is desirable in aircraft.

Finally, I might remark that none of our existing or planned reports mentions the use or the wiring of the control panels.

Dr. Korn: Do you have those figures available? I have to work a little on administration, too, in that respect.

Mr. Bisch: Every month we add to our little book several pages about new jobs; I can show you that. We make entries for every job we send to the IBM machines which involves at least a hundred man hours of engineering time. Many jobs are in the thousands of hours. We keep a close record all the time, and every two or three months I send in a progress report.

Aerodynamic Lattice Calculations Using Punched Cards

HANS KRAFT

General Electric Company

WHAT I am going to present to you is by no means a finished product. It is not even something I am exceedingly proud of. I would like to show it to you in some detail, and hope that I will receive from you some criticisms and help on how we could have this computation done in a much shorter time than it takes us now. It is possible that we are on the wrong track entirely.

The problem of the turbine engineer with all its complications is essentially as shown in Figure 1.

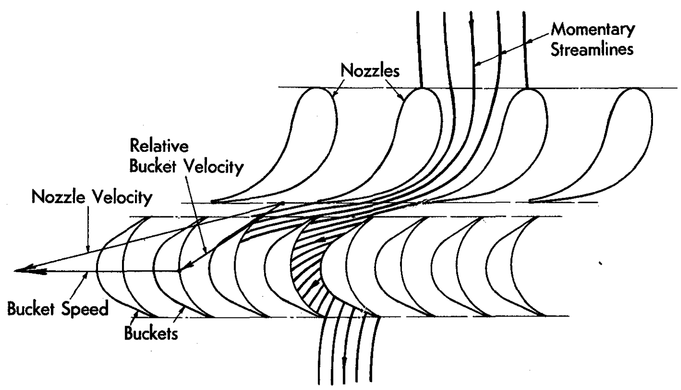

FIGURE 1

Most of you know that a turbine is essentially a windmill. A very fast flow of steam or gas issues from stationary passages which are called nozzles. The moving blades we of the General Electric Company call buckets. The steam flow streaming from the nozzles passes between them and is deflected. This process generates mechanical power which is removed by the rotating shaft.

We have had a long history of experimentation. We have experimented very intensively since 1920. We would like to do some theoretical computations in addition. We feel that we are somewhat against a blank wall with only experimental approach. It is my own honest, private opinion that further improvement in the performance of the modern turbine—and it performs very well already—will be made when, and only when, we are able to follow by calculation the flow through this nozzle and bucket combination with the buckets moving at high speed.

This means that we have to compute a flow through a row of nozzle profiles. In aerodynamic language such a row of equally spaced profiles is called a lattice. We will, in addition, need to know the flow through the bucket lattice. Furthermore, there is an interaction between these nozzles and buckets. This interaction appears as a time variation. As the buckets move past the nozzles, different configurations of the available flow space result.

We cannot rely much on the well-known approximation of the flow by that of an incompressible fluid. Our velocities are such that we always have to consider the fluid as compressible. Thus, we must first of all learn to compute compressible flow through a stationary, two-dimensional lattice; later on we must study interference between the two lattices as one passes by the other. Theoretically, we think we know more or less how to handle the problem, although as far as actual computation is concerned, we still have a long distance to go.

I should like to discuss some of the initial work which we have done to describe a simple flow through a row of buckets. It has been performed for flow of an incompressible fluid, but was done in a manner identical to that to be followed to give us the compressible counterpart of this incompressible calculation. The compressible computation still awaits the completion of a set of input functions before it can be performed.

To solve the incompressible problem Laplace's equation must be solved. We do not attempt, however, to solve a boundary value problem. We try to learn to build up profiles from given functions and accept the resulting shape if it seems to be one which we actually do want, i.e., a shape which will perform well.

We use the representation of a flux function ψ given as a function over a field with the coordinates x and y. In the compressible case we will not have this simple Laplace's equation to deal with. Instead we must solve a rather forbidding non-linear equation if the problem is written in terms of the physical coordinates x and y. To evade this situation the equation is written in other variables, those of the "hodograph" plane.

Figure 2 shows a streamline in the x-y plane. At any point along any streamline there exists a velocity vector given by magnitude and direction. The sequence of these velocities along a streamline can be represented in a coordinate system such that the ends of vectors existing along a streamline are connected. The result is a map of the true streamline. The map of all streamlines so pictured is called the hodograph representation. It describes the flow as well as does the original picture.

In the case of incompressible flow Laplace's equation describes also the field in this hodograph plane. In the case of compressible flow the differential equation applying to the hodograph plane is linear. The important consequence is that in this plane, solutions can be superimposed.

In our actual computations we are not using the true hodograph plane. We use the logarithm of the vector and

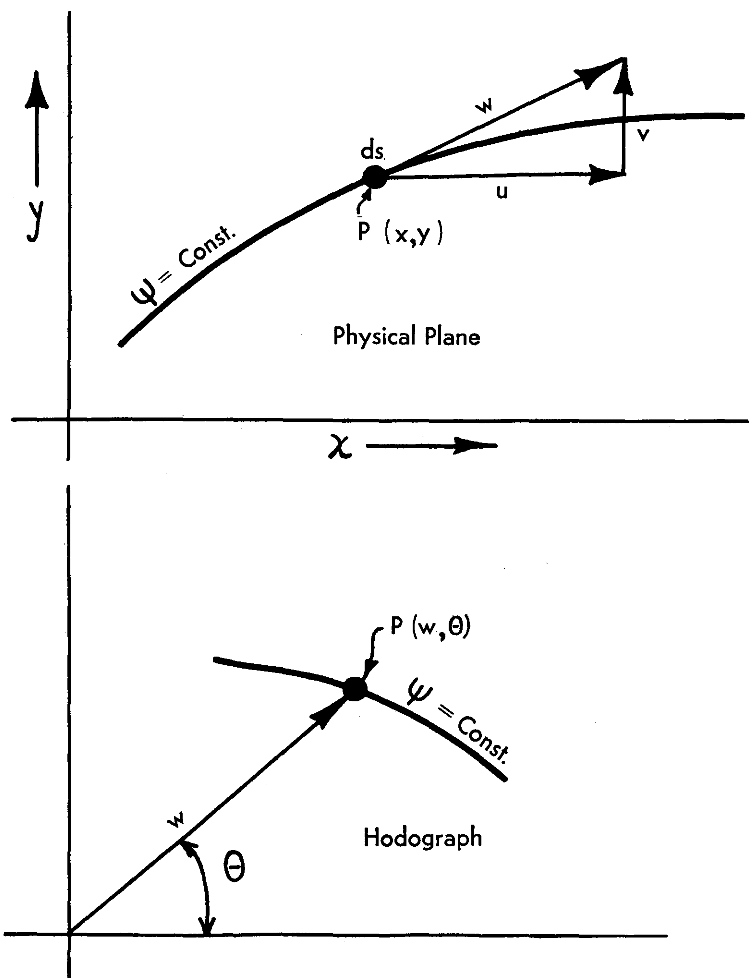

FIGURE 2

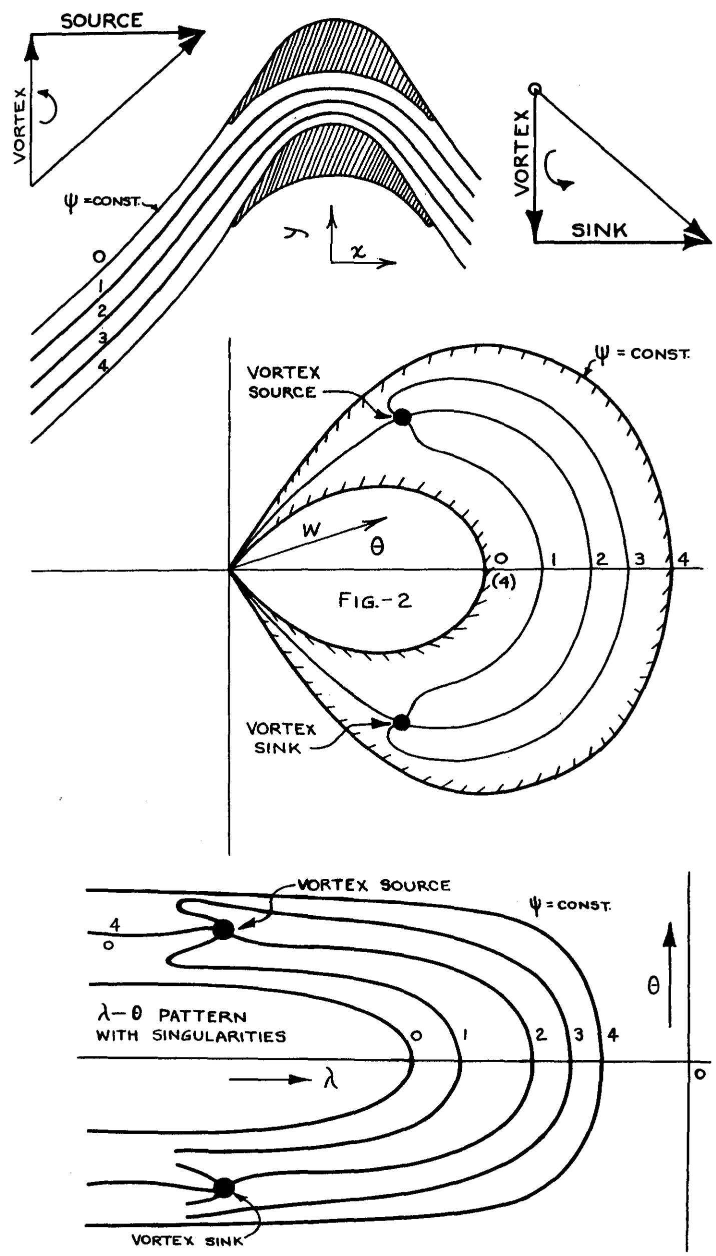

FIGURE 3

thus have a Cartesian system. Figure 3 depicts these representations for the case of relative flow through a row of turbine buckets. In the physical x-y field the flow comes from infinity to the left and disappears to infinity at the right. In the hodograph map all flow must come from the upper singularity within the closed curve and disappears into the lower singular point. A number of singularities correctly placed outside the closed curve is needed to generate this pattern. These singularities are not shown here. The streamlines within the closed curve describe all flow through the bucket lattice. The two saddle points at the left infinity of the logarithmic hodograph represent the entrance and exit of the buckets.

The object of the computation is to generate closed figures of this topology. They are generated by a vortex source representing upstream infinity and a vortex sink representing downstream infinity, combined with logarithmic singularities located outside the closed curve. Their location and strength distribution will ultimately determine the contour of the profiles.

The mathematical background of the method is shown for the more complicated case of compressible flow. The basic ingredients for the final differential equation are rather well known: continuity, irrotationality, gas law (in this case for isentropic flow). Flux function ψ and potential ϕ appear as mathematical tools.

Physical Laws

Continuity $$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

Irrotationality $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \qquad (2)$$

Gas Law (isentropic) $$\frac{p}{\rho^{\gamma}} = \text{constant} \qquad (3)$$

Mathematical Tools

Potential $$u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \psi}{\partial y} \qquad (4)$$

Stream Function $$-\rho_1 v = \frac{\partial \psi}{\partial x}, \qquad \rho_1 u = \frac{\partial \psi}{\partial y} \qquad (5)$$

Hodograph Coordinates

$$dx = \frac{d\phi}{w}\cos\theta - \frac{d\psi}{\rho_1 w}\sin\theta$$
$$dy = \frac{d\phi}{w}\sin\theta + \frac{d\psi}{\rho_1 w}\cos\theta \qquad (6)$$
$$\frac{d\phi}{w} = ds$$

Relations Between ϕ and ψ

$$\frac{\partial \phi}{\partial w} = -\frac{1}{\rho_1}\left(1 - \frac{w^2}{a_2}\right)\frac{1}{w}\frac{\partial \psi}{\partial \theta}$$
$$\frac{\partial \phi}{\partial \theta} = -\frac{1}{\rho_1} w \frac{\partial \psi}{\partial w} \qquad (7)$$

Differential Equation in Hodograph Variables

$$\frac{1 - M^2}{\rho^2_1}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{w}{\rho_1}\frac{\partial}{\partial w}\left(\frac{w}{\rho_1}\frac{\partial \psi}{\partial w}\right) = 0 \qquad (8)$$

Equations (6) show the important conversion from hodograph variables to the physical flow picture. The linear hodograph differential equation is Equation (8).

The linear fields so represented can be considered as arrived at by the intelligent addition of logarithmic singularities. Additional non-logarithmic singularities may also be added. In general, the desired results can be obtained from logarithmic singularities alone. So much for compressible flow.

Here we are considering the far simpler case of incompressible flow. It serves as a pilot process for the really desired more complicated compressible case. The difference appears in the computation of the logarithmic singularities which serve as input functions. This computation which is desperately difficult in the compressible case is much simpler for incompressible flow.

The only input function needed is given by the equation

$$F = \phi + i\psi = \ln(\ln w + i\theta), \qquad (9)$$

where

$$w^2 = u^2 + v^2 \qquad (10)$$

$$\tan\theta = v/u . \qquad (11)$$

F is the logarithmic singularity called a source. Multiplied by i it is called a vortex. This is well known to everybody reasonably familiar with conformal mapping. The source in the hodograph plane represents the axial flow component emanating from infinity of the x-y plane. The vortex furnishes the tangential component of the flow at infinity. Added together in proper proportion, i.e., with one multiplied by the correct factor to depict the flow vector at infinity, they furnish in the hodograph plane a map of the physical infinity conditions. What has been said here for the upstream infinity repeats itself at downstream infinity. Here the source is negative, in other words, a sink.

We must realize that what is desired in the end is a picture of the flow in physical space. The conversion from the hodograph map to physical flow is linear, as is clear from Equation (6). Hence x and y values can also be superimposed. If they are known over the whole field of the logarithmic singularity they can be added in exactly the same manner as can the ϕ and ψ values themselves.

In other words, if a field is composed of ϕ_1 and ψ_1, ϕ_2 and ψ_2; its physical coordinates are added from the x_1,x_2 and y_1,y_2 of the individual singularities.

We calculated this x,y field for one singularity, but we did not obtain very good accuracy (Figure 4). The equation for this field is shown on the figure. One-half of this field has been computed to great detail and accuracy by the National Bureau of Standards Computation Laboratory in New York. The field as shown is for a vortex. To find x and y for a source, the relations are

$$x_v = -y_s, \quad y_v = +x_s \qquad (12)$$

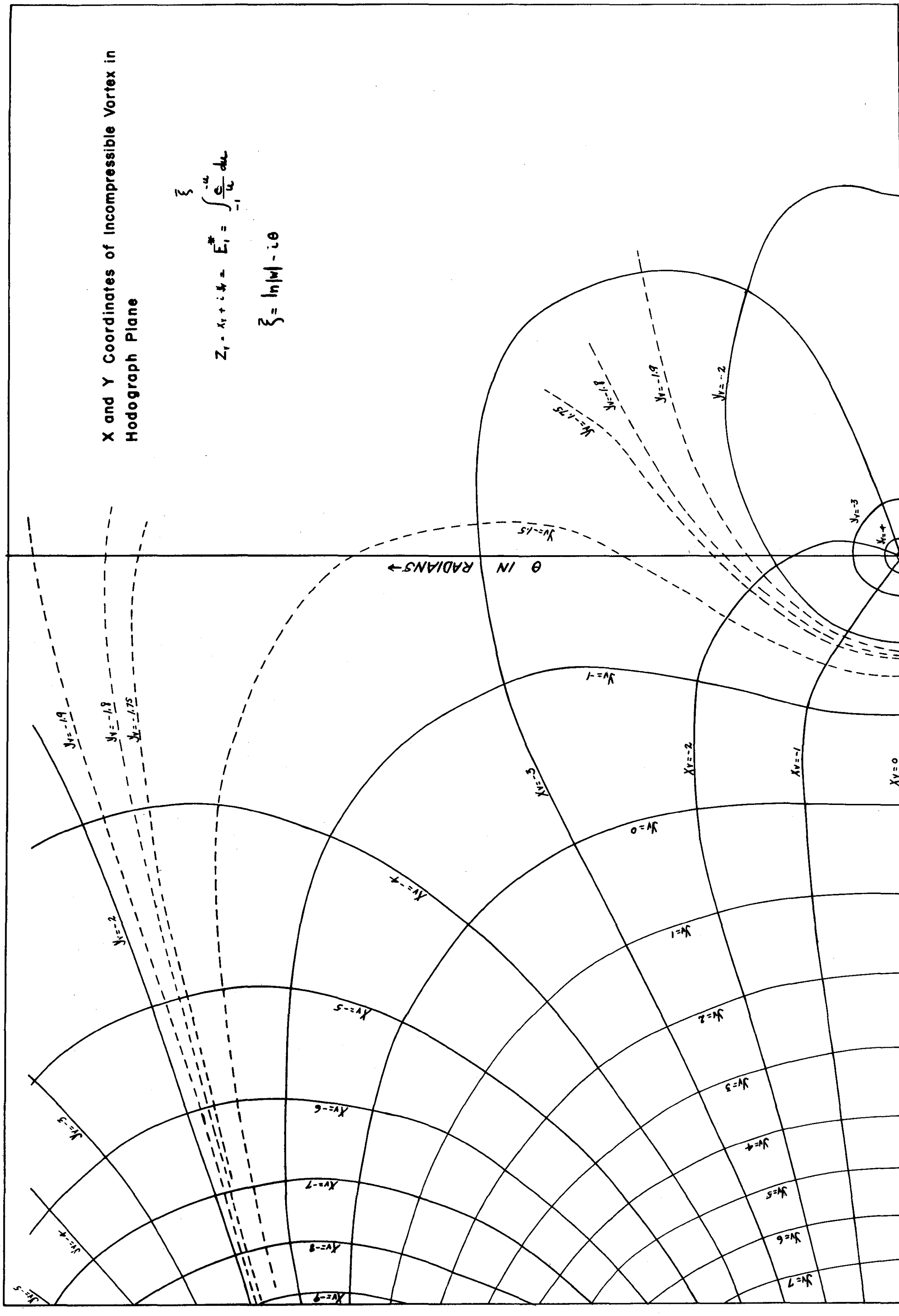
X and Y Coordinates of Incompressible Vortex in Hodograph Plane
θ IN RADIANS →
LN W →

FIGURE 4

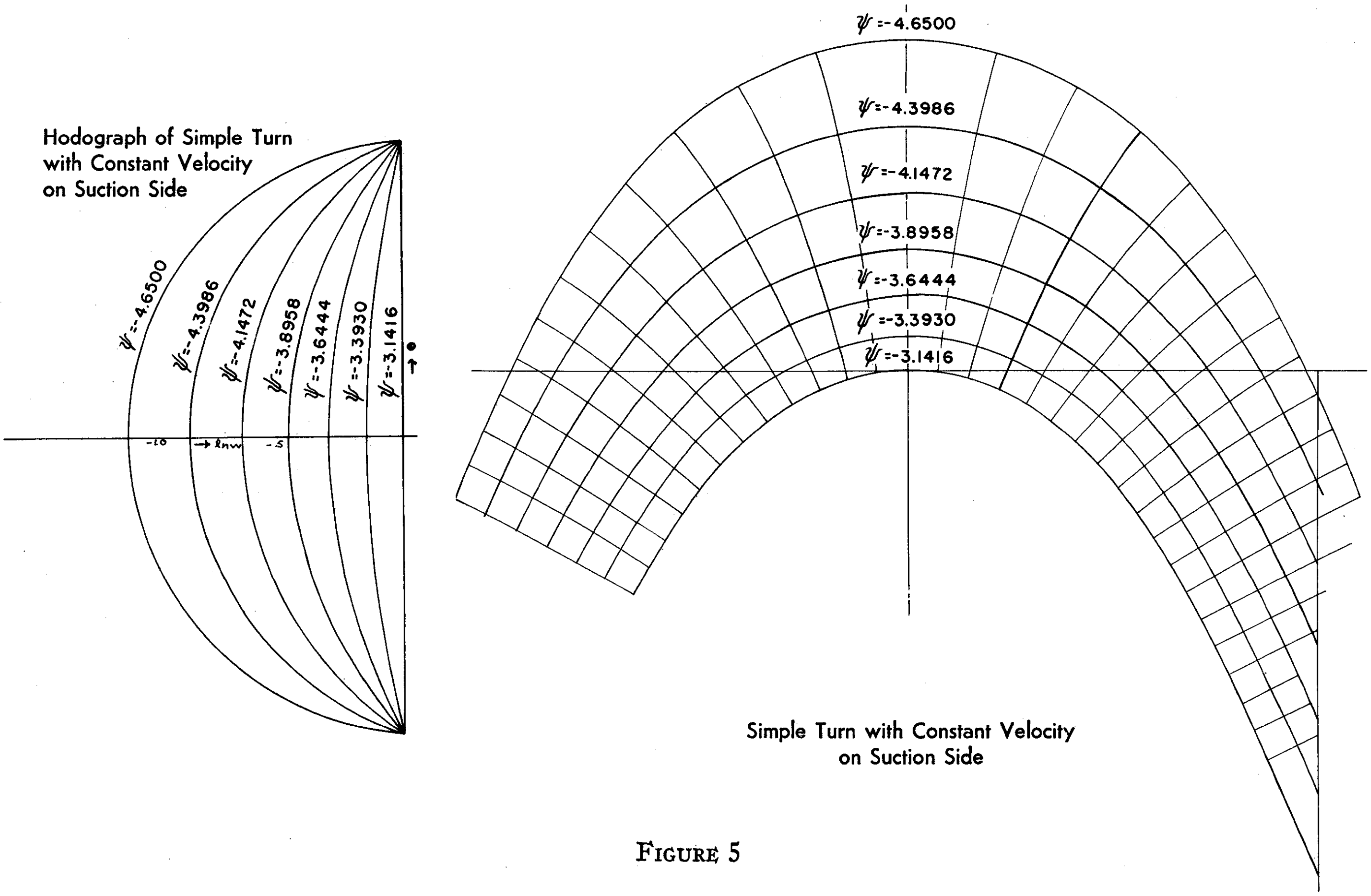

FIGURE 5

We now have four sets of numbers ϕ, ψ, x, y known at every point of the ln w-θ plane. We cover this plane by a close square mesh of numbers, always four at every intersection. This number system we can reproduce as often as needed. These systems then can be moved bodily with respect to each other. The numbers ϕ, ψ, x, y then can be added at every location and a new combined solution emerges. Obviously, it is best to move always in straight multiples of the mesh interval.

Before addition each system can be multiplied by a common multiplier. In view of what has been said before such multiplication is necessary in the case of the singularities representing the flow of infinity. Another multiplication is needed for the x-y system every time the singularity is moved. A motion in the ln w direction represents a relative shortening of physical dimensions, and one in the θ direction means a rotation of the physical system.

The equations for this correction are:

$$\bar{x} = e^{-\overline{\ln w}} (x \cos \bar{\theta} - y \sin \bar{\theta}) \tag{13}$$

$$\bar{y} = e^{-\overline{\ln w}} (y \cos \bar{\theta} + x \sin \bar{\theta}) \tag{14}$$

where $\bar{x},\bar{y}$ are the physical coordinates of the singularity displaced by ln w, $\bar{\theta}$ in the hodograph plane. These multiplications and additions must be performed for each displaced singularity before it is added to the others.

Procedure of Computation

One master stack of cards holds the four-value table. It can be reproduced onto as many stacks as singularities are needed. This reproduction can already be guided in such channels that the character of the singularity as source, sink, positive or negative vortex is taken care of. Nothing but a control of the signs is needed for this.

Next the singularities must be moved to their predetermined place. This means a displacement of the origin of each stack, in other words a change of identification. A constant is added to either independent variable. This new identification orients the singularities with respect to each other. Then we must multiply for strength and correct the x's and y's as shown above.

The field of interest for a particular computation will be smaller than that of the master table. The overflow cards are now removed. This involves one sorting.

In this manner we finish with as many stacks as there are singularities. They are properly coded with respect to each other. Next they are fed through the collator. Here all cards belonging to the same coordinates are stacked together. This new total stack enters the accounting machine where the values ϕ, ψ, x, y are added together for each coordinate. A new card is summary punched for each addition. The new resulting stack is the solution. Other singularities can be added to it if a modification of it is desired.

The solution is not yet in the form in which we need it. We must find the points for a number of (equally spaced) constant values of ψ (stream lines). The x and y values appearing along these lines furnish coordinates of the physical stream lines. One of these is the profile. What is needed is a fast and simple inverse interpolation for ψ and a direct interpolation for x and y. In the absence of a fast method, we use the old and time-honored method of cross plotting. We hope to be able some day to do this part by machine. As long as machines become faster there is hope.

Here are some of the more simple examples which we have done. We are collecting experience about the most promising combination of singularities. Figure 5 shows a very simple flow turn which has a constant velocity on the inside of the turn. By the addition of two singularities, one source and one sink, we arrive at turning streamlines, one of which has constant velocity throughout. The other streamlines decelerate first and then accelerate. None of these streamlines generates a closed profile.

The simplest closed curve we could generate is shown in Figure 6. A vortex source flows into a vortex sink and both are opposed by equal and opposite singularities. This results in a closed curve in both hodograph (circle) and physical plane. Here we have a saddle point which is not situated at minus infinity. As a result, the profile does not have a finite entrance angle at its nose. The velocity there is not zero. There is, by the way, a very simple condition which assures that if you have a closed curve in the hodograph map you will also get a closed physical flow curve.

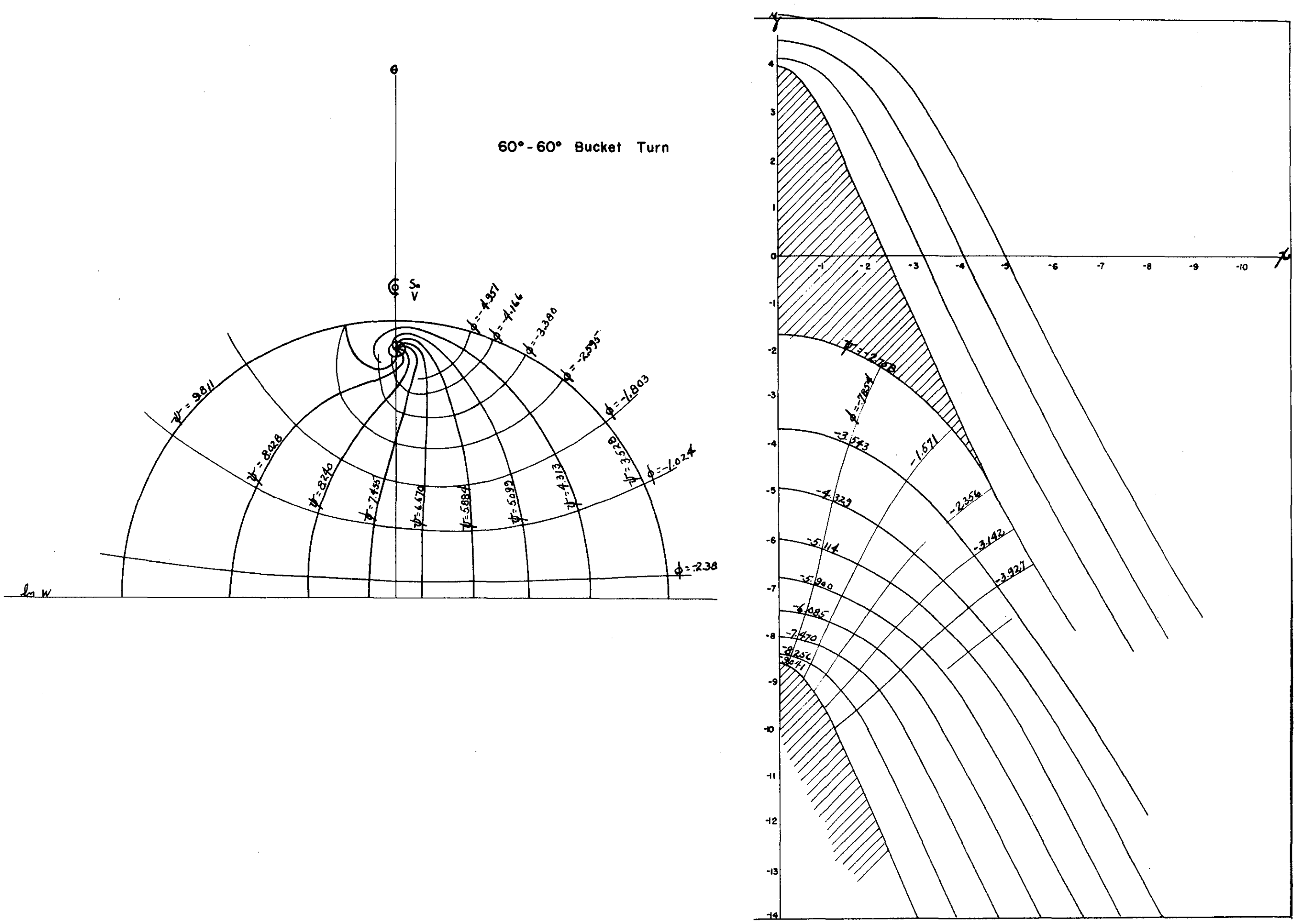

FIGURE 6

All that is necessary is that the strength of the infinity singularities matches the components of the velocity at which they are situated, and that the residue in the closed hodograph figure be zero.

I may add that the fate of the compressible counterpart of this computation is now entirely in the hands of IBM. IBM is willing to perform the very complicated calculation of the compressible singularities on the Selective Sequence Electronic Calculator. After these tables are available we can calculate subsonic compressible flow. Supersonic flow is of small interest to us. We leave that to the artillery!

Nothing will help us much, except being able to do two things. One is to compute with a great deal of accuracy. If we could sacrifice accuracy, we could proceed along cheaper ways by experiment alone. The other is to be able to mass product theoretical results, because what we primarily are after are *not* solutions for production, i.e., for the immediate turbine that goes into the shop. We want series of solutions for experimental purposes. We want to get experimental parameters which are related to blade shapes, to the interaction between blade shapes, and to a number of additional variables which now rather obscure a clear conception of the working process of a turbine. If this computation can help here we shall be able to produce a still better performing machine than we have now.

DISCUSSION

Dr. Fenn: How justified is your irrotational flow with the blades moving past each other?

Mr. Kraft: The flow can be considered completely irrotational as long as it is considered as a two-dimensional problem. Even in three dimensions, when the turbine is designed for constant circulation, you still would have an irrotational problem. It becomes rotational only when you take the boundary layer into account. This must come necessarily after we can calculate irrotationally. We have to follow the procedure which the great masters of aerodynamics have laid down. We do not think we know anything better.

Mr. Stevenson: Have you ever tried the classical aerodynamic scheme using conformal transformation?

Mr. Kraft: As I emphasized at the beginning, if it were only the incompressible solution we were after, we would not do this computation as I described it. We do it in this manner only because we can replace it by the compressible calculation as soon as the additional logarithmic singularity tables for compressibility are available. We are well aware that incompressible lattice computations can be performed by simpler procedures.

Dynamics of Elliptical Galaxies

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IN HIS CLASSICAL investigations on the dynamics of elliptical galaxies, Sir James Jean has said, "We have seen that the density of matter in the central lenticular masses of nebulae is of the order of 10^{-21} g/cm^3. The free path in a gas of this density is 10^{14} cm., whereas the diameter of the central mass of Andromeda nebula is about 1.6×10^{21} cm. . . . It follows that the various nebular configurations may legitimately be interpreted as configurations of masses of rotating gas."[1] He further calculates the free path of a star, allowing for its gravitational interactions with other stars of the cloud, and finds that for the same mean density it is equal to 10^{29} cm. (about 50,000,000 times the diameter of the nebula). He concludes that the concept of gas pressure cannot be legitimately used in connection with nebular dynamics when the nebula is supposed to be a cloud of stars, and that clouds of stars should not assume the special shapes of observed nebulae.

Today we know that the elliptical galaxies and the central bodies of the spirals are made up entirely of stars. This fact, first suggested by the stellar type of their spectra, was established beyond any doubt by the recent work of Baade[2] who was able to resolve the celestial objects into a multitude of individual stars.

To reconciliate Jeans' theoretical consideration with the observed fact, George Gamow suggests that the regular shapes of elliptical galaxies were established in some past epoch when they were entirely gaseous, and are now retained as "dead skeletons" after all the original gaseous material was condensed into stars. This suggestion agrees with the recent theory of the evolution of the expanding universe proposed by Dr. Gamow,[3] according to which the masses and sizes of galaxies can be predicted on the assumption that they originated as the result of "gravitational instability" of the expanding primordial gas. These gaseous clouds would tend to assume spherical or ellipsoidal forms depending upon the amount of angular momentum, and their internal density distribution was presumably that of the rotating isothermal gas-spheres. The linear velocities of gas-masses at each point were proportional to the distance from the rotation axis. As the star forming condensations took place under the forces of gravitation and radiation pressure, the net outward force due to the difference in gas (and radiation) pressure between the surfaces toward and away from the center of the galaxy was reduced as a result of the reduction in the volume and surface area. Consequently, the stars were accelerated toward the center under the influence of unbalanced gravitational force, assuming that at no time during the history of the galaxy do the stars exchange any appreciable amount of energy with their immediate neighbors as a result of close encounters.

The only overall forces acting on a star will be: 1. The resistance of the gas to its motion, which will be a radial force as long as the motion is radial and which will diminish in importance as the gas is consumed in the star formation process. 2. The radial force exerted by the smoothed gravitational potential of the remainder of the stars. Then, the stars will tend to oscillate radially through the center of the galaxy. The amplitude of oscillation or the points of maximum elongation of the newly acquired elongated elliptical orbits of the stars should correspond to the distances at which they were originally formed. As more and more stars were formed, the major axes of original elliptical orbits were gradually changing due to the gravitational action of other stars which have originated outside of them, but are now penetrating during a certain fraction of their motion into the interior regions. As a result of these interactions, the stars acquired an altered distribution. It is expected, however, that the stellar orbits were not shuffled; i.e., the stars which were formed at larger distances from the center and therefore with larger angular moments will also have their points of maximum elongation farther out from the center. In observing mean radial velocity of stars at different distances from the original rotation axis, with respect to an observer, we will essentially obtain the tangential velocities of those stars which pass through their point of maximum elongation at that distance. Since the stars which move beyond that point will be much smaller in number and will also present us only with the projection of their actual velocity, it can be expected that the observed rotational velocities will increase with the distance from the center (linearly in the first approximation), giving the impression that the entire galaxy is rotating approximately as a solid body. This "solid body" rotation is actually observed in elliptical galaxies and in the central bodies of spirals, and was considered an unexplainable phenomena in view of the large free passes of individual stars.

In the present work we plan to analyze the observed density distribution in the elliptical galaxies which was very carefully measured by Dr. Hubble.[4]

Computation of the initial density distribution in a spiral galaxy before formation of stars:

Let $\rho(r)dr$ = the present mass in the shell between radii r and $r + dr$

$\beta(a)da$ = the initial mass between the shells of radii a and $a + da$

$P(r)$ = the present potential at r.

A stellar mass starting at a and dropping inward to r will lose the potential energy

$$m[P(a) - P(r)] = mv^2/2$$

and hence acquire the velocity

$$v = \sqrt{2[P(a) - P(r)]}\,.$$

The time which the star will spend in the shell of thickness dr will be dr/v (on the inward trip), and the fraction of all its time which it spends in this shell will be $dr/vT(a)$, where $T(a)$ is the time required for the mass to drop from a to the center. The contribution to the mass observed now between r and $r + dr$ arising from the mass which started to drop from between a and $a + da$ is

$$\frac{\beta(a)\,da\,dr}{T(a)v}\,.$$

Then the total mass between r and $r + dr$ is

$$\rho(r)dr = \int_r^R \frac{\beta(a)da\,dr}{T(a)v}\,.$$

Therefore

$$\rho(r) = \frac{1}{\sqrt{2}}\int_r^R \frac{\beta(a)\,da}{T(a)\,[P(a) - P(r)]^{1/2}}\,.$$

Let $A = \log_{10}a \qquad \frac{dA}{da} = M\cdot\frac{1}{a}$

$Q = \log_{10}r \qquad \frac{dQ}{dr} = M\cdot\frac{1}{r}$

M = modulus = .434

then $\beta(a) =$

$$\frac{2\sqrt{2}}{\pi}\cdot\frac{M}{a}\,T(a)\,\frac{d}{dA}\left[\int_A^{\log R}\sqrt{P(r) - P(a)}\,\frac{d\rho(Q)}{dQ}\,dQ\right]. \quad (1)$$

For integration, the Gregory formula is suggested, since the errors of integration are readily estimated:

$$\frac{1}{w}\int_a^{a+nw} f(x)dx = \left(\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{n-1} + \frac{1}{2}f_n\right) - \frac{1}{12}(\Delta^{i}f_{n-1} - \Delta^{i}f_0) - \frac{1}{24}(\Delta^{ii}f_{n-2} - \Delta^{ii}f_0) - \frac{19}{720}(\Delta^{iii}f_{n-3} - \Delta^{iii}f_0) - \ldots$$

For obtaining derivatives near the end of the tabular sequence we use the following:

$$wf'(a) = \Delta^{i}f(a) - \frac{1}{2}\Delta^{ii}f(a) + \frac{1}{3}\Delta^{iii}f(a) - \frac{1}{4}\Delta^{iv}f(a) + \ldots$$

Away from the ends of the tabular sequence the following central difference formula converges more rapidly:

$$wf'(a) = \frac{1}{2}(\Delta^{i}a_{\,1/2} + \Delta^{i}a_{1/2}) - \frac{1}{12}(\Delta^{iii}a_{\,1/2} + \Delta^{iii}a_{1/2}) + \frac{1}{60}(\Delta^{v}a_{-1/2} + \Delta^{v}a_{1/2}) - \frac{1}{280}(\Delta^{vii}a_{-1/2} + \Delta^{vii}a_{1/2}) + \ldots$$

By definition

$$T(a) = \int_a^0 \frac{dr}{v} = \frac{1}{\sqrt{2}}\int_a^0 \frac{dr}{[P(a) - P(r)]^{1/2}}$$

$$Q = \log_{10}r;\ \frac{dr}{dQ} = \frac{1}{dQ/dr} = \frac{r}{M} = \frac{r}{.434}$$

$$T(a) = \frac{1}{M\sqrt{2}}\int_A^{-a} \frac{r\,dQ}{[P(a) - P(r)]^{1/2}} \quad (2)$$

Near $r = a$, $P(r)$ becomes nearly equal to $P(a)$ and the integrand diverges. To overcome this a series approximation in the neighborhood of $r = a$ is suggested.

Let $h = (a - r)$. Hence the first term is

$$a\sqrt{\frac{2}{M\,a\,X}}\,\sqrt{h} \quad . \tag{3}$$

For small a the first term approaches zero, and in the second we may set $3Ma = 4\pi a^3 \rho\ (0)$, and $\rho(a) = \rho(0)$.

Let $h = a$; then to the first approximation

$$T(a) = \frac{1}{4}\sqrt{\frac{1}{6\pi G\,\rho(0)}}\ . \tag{4}$$

The third term of the series should be calculated if greater accuracy is desired.

The equations (1), (2), (3) and (4) with which we are here concerned, lend themselves readily and very conveniently to solutions on standard IBM equipment. It is gratifying that a computing laboratory of this nature is available for these investigations. Although to date no conclusive results are available, a more thorough investigation of the problem is possible. It is the hope of the authors that in the near future they will be in a position to publish results of these investigations which will conclusively decide whether this theory is valid.

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DISCUSSION

Mr. Hollander: Do we have any proof that stars are moving radially, either in or out?

Mr. Belzer: It would take very many years to notice any motion in the stars. I think that according to Dr. Gamow's theory of the evolution of the universe, in a period of about 200,000,000 years the stars might have made seven or eight oscillations.

Application of Punched Cards in Physical Chemistry

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THE FOLLOWING concepts of the application of punched card machines are the consequence of the particular problems in which the author is interested, but probably the examples are typical of present-day theoretical and physical chemistry.

The Hollerith machine was invented for scientific work, but reached the present stage of refinement because of its application to commerce, and in the field of chemistry it will return to science because of its bookkeeping facilities. The chemical literature has reached such an enormous volume that the tremendous burden of bookkeeping is a serious detriment to efficient research. The application of punched card techniques to indexing is the center of interest in chemistry today, and outweighs the other possibly more important applications. One is the recording, storage and handling of experimental data. Next, there is as a rule an opportunity for machine methods in the calibrating and correcting of these data. After this there is the analysis of the data, various correlations to be searched for and other statistics to be extracted. Finally, there are purely theoretical calculations.

Recording Experimental Data Digitally

Today, the recording of data is extremely widespread. Almost always continuous variables are used to measure quantities in an experiment or industrial process; the most primitive is the visual reading of a meter scale, and recording of the nearest number on the scale. Recording potentiometers are commonplace. Photographs of oscilloscope patterns are used in transient phenomena. Such records are rarely useful in themselves. Even if the primary data on the record is sufficient by itself (for example, the temperature recorded as a function of time), the study of more than a few dozen charts by eye is almost impossible. Usually the primary recorded data have to be converted to some absolute quantities. An example is the recording of infrared spectra. The actual conventional record consists of the distance a pointer is deflected for various angular displacements of the paper roll. From a knowledge of the parameters of the instrument the deflection can be converted to an absorption coefficient and the distance along the paper converted to wavenumbers. The absorption coefficients at a given wavenumber are invariant to the particular experimental conditions; it is these quantities, not the original record, which have universal interest. This conversion and calibration can only be done after a first step of visual "reading off" the charts, point by point. This is a process of conversion from a continuous physical variable to an abstract digital number. The conversion and calibration now proceeds by digital calculations. Clearly, it would be most desirable to record the original physical variable (often a voltage) as a digital number. At first sight, it might seem that such a process is less accurate. However, this is what the eye ultimately does in "reading off" the chart. Furthermore, every good experiment should be done with equipment that has a definite "noise" level (in a general sense) just visible. Even measuring a distance should be done with a scale, or a micrometer eyepiece, or a cathetometer whose accuracy is just sufficient to meet the requirements, or if a limiting factor, at its extreme of sensitivity, so that each reading has a few percent of "noise." The magnitude of the noise can be taken as the unit in the digital scale; and a digital reading is as accurate as the experiment, even though the continuous trace may look more precise.

If the readings are to be made digitally, the binary system is the best. First, a punched card works on a binary system. There are two and only two "digits"—a hole or no hole. Secondly, the binary system is most economical in number of required digits per number (above eight). Finally, as a practical matter only one punching time is required to (multiply) punch a binary number <1024 in a single column of a standard card, in contrast with the four punching times for a four-digit decimal number.

The problem of recording data has been solved by a Digital Reader, which reads a voltage, converts it to a digital binary number and punches this in a card. An accuracy of one percent has been achieved with a very simple circuit. The resulting seven digit (<128) binary number is punched in one column of a card at the fastest punching speed of IBM equipment, namely twenty-five a second.

Processing of Data

With the original measurements directly punched on cards as primary data with the original "noise" retained, the conventional processing of the data can proceed with machine methods. The example of infrared spectra brings out the power of punched card methods in extracting more information from experimental data than can be done by continuous record. Hundreds of new spectra are being taken every day. For the most part they are compared with spectra taken in the same laboratory, and in almost all cases, with spectra already extant in the literature. Several thousand infrared spectra are now generally available, but their comparison is a difficult matter. In addition to the various scales used in reducing drawings for publication, there are several conventions of plotting wavelength or wavenumber, increasing or decreasing to the right. Finally, even with two records on the same scale, there are some rather subtle comparisons to make. In general, the problem is not whether two spectra match at every point, but whether they have certain peaks in common, and to what extent. This is similar to the problem of weather prediction carried out by punched card methods, based on the matching, within tolerances, of today's weather map with a file of the last forty years' maps. This is an ideal situation for a collator. Methods of calibrating the data, and extraction of statistical information, are relatively conventional and are to be found in the literature.

There is a converse of the above problem, namely the plotting of results of calculations done on cards. There is no question that a plot is more readily understandable than a table of values. It would be very desirable to have an instrument which would take the impulses from an accounting machine and convert them to a continuous quantity, presumably a voltage, which would drive a recorder. Lacking such equipment, we have derived a method of using the type bars of a 405 to plot points.[1]

Theoretical Calculations

At first sight, one might be sure there are many opportunities for punched card methods in theoretical chemistry. There are, however, several arguments against their use. The cost, if not the very presence, of a battery of IBM machines in a chemical laboratory will have an undue influence on the type of work done there. There will be emphasis on the ponderous data collecting and large scale poring over of material notable for its quantity rather than quality, away from the elegant simple experiment. On a more theoretical level, the interests and, ultimately, training of the research worker will be away from analysis and be likely to get in a rut of conventional punched card approaches. There is no time to read Whittaker and Watson when machines stand idle!

Another difficulty in the introduction of standard machine methods is the fact that IBM equipment is a parallel type of computer. The scientific approach inherited from pre-machine days is sequential. For example, many problems are developed with the aim of finding the solution as the root of a polynomial. Now there are no analytical ways of finding roots of polynomials. But IBM machines are not suitable for the numerical solution of a single polynomial, because they are efficient in parallel and not in sequential calculations. If, however, the problem, perhaps by generalization, requires the solution of a few thousand polynomials, the process could be run in parallel even though it required very many sequential steps. The best known numerical method of solving polynomials involves continued fractions, which are known to converge, and hence behave better than many approaches in which the root has to be raised to a high power. The recent development of the 602 and 604 which can divide has opened up the field of extraction of roots to machine methods.

The repetitive feature of punched card machines is a possible advantage to physical chemists, especially in the construction and tabulation of functions. One example would be the tabulation of the free energies of all substances for which information is available, from empirical constants. This could be done to great advantage at every degree from $-273°$C. to $5000°$C. and thus save a great deal of interpolation. Another application is the direct calculation of the thermodynamic functions from fundamental frequencies from spectroscopic analysis, without the forcing of such data into simple empirical formulas in order to sum or integrate.

The purely theoretical problems of chemistry lie in a field in which the role of machines, although certain, is quite obscure. Chemists deal with molecules, and except for hydrogen, a rigorous quantum mechanical approach is beyond the power even of the most advanced machines today. One difficulty with the quantum mechanics of a molecule is that it is many-dimensional, and integration even in three dimensions in general involves astronomical numbers of unit operations. It may be possible to approach these problems with some sort of statistics, as Dr. Thomas did so well for atoms. Another difficulty in theoretical chemistry, as contrasted with fundamental physics perhaps, is that the solutions of problems do not involve analytical functions. This appears even in the simplest problems, such as the interpretation of infrared spectra. The spectrum of water, for example, consists of several thousand lines which are differences of a fewer number of energy levels. The latter, however, are not spaced according to any elementary function.

They are roots of polynomials, and are non-elementary functions of the moments of inertia, so can only be expressed as tables. Thus, advanced spectrum analysis can only be approached by machine methods.

Still a further difficulty arises in the real problems of physical chemistry from the nature of experimental data themselves. The unravelling of the rotational structure of an infrared spectrum has been described above as being very complex. One might infer that, with machine methods and enough time, a unique interpretation could be made. This is true when the spectrum is completely resolved, as in the photographic infrared. In the main infrared region, lines are not completely separated. The spectrum consists of a continuous record consisting of a number of peaks, each of which contains a number of more or less resolved "lines" which are not mathematical lines, but Gaussian curves. Thus, the conventional analysis by finding a set of constant differences between the various lines fails, because the overlapping of lines in peaks leads to uncertainties in their position of the order of magnitude of significant differences in the differences.

There is an analogous situation in many x-ray and electron-diffraction spectra. The classical approach of deducing a unique structure from its observational data cannot be carried through. Modern spectrum analysis almost always is achieved by the stochastic process. A structure of the molecule is assumed. Certain theoretical expressions are used to calculate the appearance of the spectrum in which such a structure would result. The calculated spectrum is compared with the observed. Successive trial structures are assumed until a satisfactory agreement between calculated and observed spectra is achieved. There is no proof the final fit is unique, but as a rule the observed spectra have sufficient complexity that the probability of any other structure giving rise to the same or better fit is remote. The application of the stochastic method to the analysis of rotational structure of infrared spectra has been described in detail elsewhere.[2]

It is appropriate to review briefly the wiring of the various IBM machines in this work.* Many of the calculations are straightforword. The only unusual feature in the collator is the effective use of splitting up the comparing magnets into nine fields. This allowed comparing, and hence merging on three fields of quantum numbers simultaneously, two of which increased and one decreased.

The whole process consisted of forty steps, of which seventeen were done on the accounting machine. It might be interesting to point out that all calculations were done on the same control panel. The most noteworthy single calculation done on this control panel was the calculation of the expected spectrum with a finite slit. A deck of cards, one for each small absorption line in the spectral region ($300\,cm^{-1}$ wide) is made as the final step in the purely theoretical treatment. Each line is represented by a card giving its position in wavenumbers and its intensity. These cards are summary punched over a small interval ($0.5\,cm^{-1}$) of the spectrum, so that lines less than 0.5 wavenumbers apart are combined in intervals. This deck of summary cards is made complete with blanks, to give a final deck, a card appearing for every $0.5\,cm^{-1}$ interval in the whole region. This deck then represents the appearance of the spectrum if the resolving power were $0.5\,cm^{-1}$. In actuality it is higher. A region of wavenumbers is seen by the slit of the spectrometer at any one time. It is desirable then to compute the actual transmission at each wavenumber ν

$$T(\nu) = \sum_{\sigma=-2}^{+2} \rho_\sigma \cdot I\,(\nu + \sigma)\,,$$

where $\rho_{-2}, \rho_{-1}, \rho_0, \rho_1, \rho_2$, are weight factors describing how much on either side of ν the slit sees. This summation is to be carried out for each ν, proceeding by intervals of $0.5\,cm^{-1}$ along the region of $300\,cm^{-1}$. It was found that values of $\rho = 1,2,4,2,1$ or $1,2,1$ satisfactorily expressed the experimental slit shape function. Summary cards $T(\nu)$ were obtained in one pass through the accounting machine, using progressive totals in five counters, where counter entries were fed the number $I(\nu)$ by a permuting switching arrangement. A minor total cycle occurred every card. One counter has to be cleared every card. This was done connecting the counter exit to entry through the permuting class selectors.

It should be mentioned that some calculations were made possible on the single accounting machine control panel by the use of external manually controlled gang switches which connected different fields of the cards via brushes to different counters, which were too few in number. The well-known "Octal" tube socket was used for additional hubs, the standard IBM wire fitting the holes in these sockets somewhat better than the hubs in the control panel!

Another large field in physical chemistry where large scale computing machinery will play a great role is statistical mechanics. The properties of a single molecule have to be determined from experimental measurements by a stochastic process as outlined above. But the macroscopic behavior of materials is an average of certain quantities of an extremely large number of molecules, each one moving, vibrating, rotating with different velocities, or in a different configuration. A very simple example is the theory of rubber-like elasticity as based on a simple theory of the statistical mechanics of high polymers. Rubber molecules are long chains of ten thousand or more seg-

*Complete wiring diagrams will be presented in a report to the Office of Naval Research who supported some of the spectrum analysis with punched card equipment.

ments. A molecule can therefore exist in very many different configurations, and as a consequence has high entropy. On stretching, the molecules become less random, and the chains tend to become parallel. The decrease in entropy is responsible for the strong force in a piece of stretched rubber which attempts to contract it to its original state of maximum randomness. Clearly the force, and hence the modulus of elasticity of a plastic made of long chain molecules, could be calculated if we could enumerate the possible configurations of such an assembly of chains. This can be done in a simple way and leads to results in rough agreement with experiments, at least for highly elastic substances. The simple theory has a number of defects, principally the fact the molecules have volume and can only occupy the same volume of space once. The problem of this effect of "excluded volume" on the number of configurations is a topological one of great difficulty. We have set out to solve this topological problem by a straightforward enumeration of the configurations of chains. In essence, we have investigated the famous "random walk" problem, in a tetrahedral lattice accounting for the effect of excluded volume.

This, then, is an example of the use of punched cards in sampling a Gibbsian ensemble, in which each system is described appropriately on a set of cards. The required statistical averages can be very readily made by arithmetical means by conventional processing of these samples on cards.

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DISCUSSION

Mr. Bell: What mechanical or electrical system is between your prism and the punch?

Dr. King: A whole bunch of relays; it took me a day to wire them up. The method is not very complicated.

Dr. Caldwell: This problem of converting a voltage to a number on punched cards is going to occur more frequently and in more difficult form than is described here. One case involves the fact that the voltage is coming from a device over which you have no control. I think the general approach is that you can apply the pulse code modulation system. Within the last year and a half, the Bell Technical Journal has contained several papers on that subject, the general principle being to compare voltage patterns against a standard voltage. That means that you can easily produce these binary digits; the real problem is that they are coming too fast for any punch to record directly.

Dr. King: We don't have a storage problem because we get all the digits out more or less simultaneously. We take a new reading as soon as the old one is punched.

Dr. Grosch: May I make a remark about this question of finding the roots of a polynomial? There is a class of problems which can be handled in parallel fashion—polynomials of degree, say, five to twelve. If you have several to do at once, so much the better. The process is to evaluate the nth degree polynomial for $n + 1$ equal spaced values of the variable x, *without* rounding. This is a severe limitation, since it involves large multiplications toward the end; the answers will usually be about $n + 6$ significant figures for $n < 20$. These exact answers are then differenced n times; the single nth difference should equal $n!\omega^n$ times the original coefficient of x^n, where ω is the interval; this is used as a check. Further values of the polynomials can then be built up from the constant nth difference on the 405 three or four orders at a time, with summary punched intermediate results. The tabulation of the polynomial can be extended to cover all real roots, or restricted to the neighborhood of a single root. Inverse interpolation for the exact root is the final step; usually this is hand work.

If the above procedure seems too complicated, another trick worth trying on the 602 or 604 is to prepare a deck of $n + 1$ cards carrying the coefficients of the polynomial, highest degree coefficient first. A card carrying a guess at the root is placed in front of the deck, and one pass through the calculating punch evaluates the polynomial. A new guess is made, key punched on a fresh card, and the process repeated. The 602 or 604 is here used as a specialized desk calculator.

*Application of Punched Card Methods to the Computation of Thermodynamic Properties of Gases from Spectra**

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THE PRINCIPLES of the statistical calculation of thermal functions of gases from spectroscopic data have been used for many years with a large degree of success. The basis for much of this work is the so-called "summation method" which was first used by Hicks and Mitchell.[1]

From the distribution of the molecules of a gas among the various energy levels of the molecule, the energy and other thermal properties of the gas can be found through the evaluation of the partition functions. In the summation method, this summing takes place over the actual energy levels of the gas. When there are large numbers of energy levels, as in all but the simplest molecules, the direct application of this method becomes impractical and labor saving devices must be used. In the past, these have taken the form of substitution of integrals for the series involved as, for example, in the extended Mulholland treatment.[2] The present paper introduces a method whereby the original summation method is adapted to use of punched cards.

The energy levels of the molecule may be obtained through a study of the band spectrum of the gas. In the equations for the thermodynamic properties for the gas from these energy levels, the contributions due to translation may be separated from the effects due to rotation and vibration, thus simplifying the expressions to be evaluated. Johnston and Chapman[3] have introduced the following notation whereby the contributions of the rotational and vibrational energy of the molecule may be expressed simply in terms of three basic quantities:

$$\Sigma A = \sum_i p_i \, e^{-\epsilon_i/kT} \tag{1a}$$

$$\Sigma B = \sum_i p_i \, \epsilon_i \, e^{-\epsilon_i/kT} \tag{1b}$$

$$\Sigma C = \sum_i p_i \, \epsilon_i^2 \, e^{-\epsilon_i/kT} \;. \tag{1c}$$

For convenience, these three quantities may be redefined in terms of base ten exponentials:

$$\Sigma A^* = \sum_i p_i \, 10^{-\epsilon_i/zkT} \tag{2a}$$

$$\Sigma B^* = \sum_i p_i \, \epsilon_i/zkT \cdot 10^{-\epsilon_i/zkT} \tag{2b}$$

$$\Sigma C^* = \sum_i p_i \, \epsilon_i^2/zkT \cdot 10^{-\epsilon_i/zkT} \;. \tag{2c}$$

In the above equations ϵ_i is the energy of the molecule in the ith quantum state, p_i the statistical weight of this state, k the Boltzmann constant, T the absolute temperature of the gas, and z the numerical constant ln 10.

The thermodynamic functions are given by the following equations:

$$E^0 - E_0^0 = 3/2\, RT + zRT\, \Sigma B^*/\Sigma A^* \tag{3a}$$

$$C_p^0 = 5/2\, R + z^2R\, [\Sigma C^*/\Sigma A^* - (\Sigma B^*/\Sigma A^*)^2] \tag{3b}$$

$$S^0 = 3/2\, zR \log M + 5/2\, zR \log T - 2.3140 + zR\, [\log \Sigma A^* - \Sigma B^*/\Sigma A^*] \tag{3c}$$

$$F^0 - E_0^0 = 5/2\, R - 3/2\, zR \log M - 5/2\, zR \log T + 2.3140 - zR \log \Sigma A^*. \tag{3d}$$

The final term in each of the equations (3) gives the contribution of the rotational and vibrational states, while the preceding terms are due to the translation of the molecules. New quantities appearing in these equations are the gas constant R, and the molecular weight of the gas M.

This laboratory, with the assistance of Dr. Thomas of the Watson Scientific Computing Laboratory, has set up a punched card method for obtaining the quantities ΣA^*, ΣB^*, and ΣC^*, once a punched card table of the energy

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levels is available. These are readily computed in many cases. Since the sums are to be computed for a number of different temperatures, it becomes convenient to transform the exponentials of equations (2) into double exponentials. The sums A^*, B^*, and C^* now have the form

$$\Sigma A^* = \sum_i p_i \, 10^{-10^{x_i}} \quad (4a)$$

$$\Sigma B^* = \sum_i p_i \, 10^{x_i} \, 10^{-10^{x_i}} \quad (4b)$$

$$\Sigma C^* = \sum_i p_i \, 10^{2x_i} \, 10^{-10^{x_i}} \quad (4c)$$

where $x_i = \log \epsilon_i/zk - \log T = \zeta_i - \log T$. The advantage of this will be apparent when the calculations of these sums at different temperature is discussed. For the present, the temperature will be set at 1°K., so that $x_i = \zeta_i$. A master deck of the functions 10^{-10^x}, $10^x \, 10^{-10^x}$, $10^{2x} \, 10^{-10^x}$ was prepared for the argument $x = -8.00 \, (.02) + 1.00$. This set includes all the significant values of the three functions (which vary between zero and one). Before this table can be utilized, it is first necessary to obtain the logarithms of the quantities ϵ_i/zk and reduce these to the arguments of the master table. To achieve the former of these objects, a six place optimum interval logarithm table is used, whereby x_i is obtained to six decimal places and one whole number. Four point Lagrangian interpolation coefficients[4] are used to distribute the statistical weights p_i of the energy levels ϵ_i to the appropriate arguments of the master deck.

The punched card Lagrangian interpolation table built up for this purpose consists of 5000 cards at intervals of 0.0001 in the argument, $p = \triangle x/h$, for the range 0.0001-0.5000. Since the table reflects itself for the range 0.5000-1.0000, the table is punched in a symmetrical manner around the center of the card, with the right half reading in reverse from right to left. Tumbling the cards puts them in position for use in the upper half of the interval range. A secondary argument, $q = 2p$, is also punched on the table so that the table may be collated directly with the last four digits of x_i. To accomplish this most readily, the detail cards are first divided into odd and even groups on the second decimal digit of x_i to determine which half of the table is to be used.

The use of the interpolation coefficients is shown most readily by the following illustration. Suppose that for an energy level ϵ_i of statistical weight p_i, the log ϵ_i/zk is equal to 0.498306. The distribution of the statistical weight among the four adjacent arguments is given in Table I.

TABLE I

$x_{n-1} = 0.46$	$p'_{n-1} = p_i A_{-1}$
$x_n = 0.48$	$p'_n = p_i A_0$
$x_{n+1} = 0.50$	$p'_{n+1} = p_i A_1$
$x_{n+2} = 0.52$	$p'_{n+2} = p_i A_2$

The four coefficients, A_{-1}, A_0, A_1, and A_2 are those corresponding to the argument $q = 1.8306$. A prepared table of x_{n-1}, x_n, x_{n+1} and x_{n+2} with the argument x_n is also reproduced onto these same detail cards so that the proper argument may be matched with the new weights p'_n. In general, there will be a considerable amount of overlapping, and so the p'_n's are next summed for each value of x_n such that a total weight, $w_n = \Sigma p'_n$, is obtained for each different value of x_n. In practice, it has been found most convenient to make up three additional sets of cards so that the arguments x_n and the products $p'_n = p_i A$ always appear in the same fields and the new weights are obtained easily on the accounting machine.

The advantages of the procedure described above are twofold. In the first place, there is a great reduction in the total number of terms over which the sums are to be taken. Even in a relatively simple gas, the total number of thermodynamically important energy levels lies in the thousands, whereas the master deck of functions for these new arguments contains at most 400 cards. Secondly, the functions 10^{-10^x}, $10^x \, 10^{-10^x}$, $10^{2x} \, 10^{-10^x}$ for the arguments x at intervals of 0.02 are available in a permanent file and need only be reproduced into the detail cards to be multiplied and summed.

Up to this point, the temperature has been neglected. However, by shifting the table cards one value (0.02) of x, the detail cards may be prepared for a new temperature. The entire temperature range may thus be covered in this way in logarithmic increments of 0.02 in T. This is the major advantage of the double exponential form adopted for the problem. The range of log T from 0.00 to 3.84 is sufficient to cover all temperatures up to 6000°K. The manipulation of the sets of cards to take account of the temperature shifts is more clearly shown by the following numerical example, taken from an actual calculation. The symbol ζ_n has been introduced for the x_n used earlier to avoid confusion. The smallest value of ζ_n is 0.24, and the first set of cards prepared is that for the highest value of log T which is equal to 3.84 if the computations are to be carried up to 6000°K. Successive cards in this set will have arguments and weights as given in Table II.

TABLE II

w_1	$\zeta_1 = 0.24$	$x_1 = 0.24 - 3.84 = -3.60 = \bar{4}.40$
w_2	$\zeta_2 = 0.26$	$x_2 = 0.26 - 3.84 = -3.58 = \bar{4}.42$
w_3	$\zeta_3 = 0.28$	$x_3 = 0.28 - 3.84 = -3.56 = \bar{4}.44$, etc.

The last card in the set should have the argument $x = 1.00$. At this point, the functions have dropped to zero and any left over cards may be discarded.

The second set of cards will be for the next lower temperature, log $T = 3.82$. The master set is shifted up one value, thus w and the x_2 are matched. The last value of w_n will not match any table card and may be discarded. This is shown in Table III.

TABLE III

$$w_1 \quad \zeta_1 = 0.24 \quad x_2 = 0.24 - 3.82 = -3.58 = \overline{4}.42$$

$$w_2 \quad \zeta_2 = 0.26 \quad x_3 = 0.26 - 3.82 = -3.56 = \overline{4}.44$$

$$w_3 \quad \zeta_3 = 0.28 \quad x_4 = 0.28 - 3.82 = -3.54 = \overline{4}.46, \text{ etc.}$$

The process is repeated until the entire temperature range has been covered. It has been found possible to condense the number of individual temperature sets to one third by including all the necessary data for three temperatures on one set. The sums A*, B*, and C* may now be obtained according to equations (4). Since the individual products were not required, this was done by progressive digiting. There may be as many as thirteen digits in w_n, so that this is quite a prodigious task. It is particularly unpleasant to handle since the individual groups are so small.

Once the sums have been obtained for the arbitrary temperatures, they in turn are interpolated using the four point coefficients, yielding the results for the desired temperatures. From these, the thermodynamic properties may be readily computed by using the relations given in equations (3).

A comparison of the hand and machine computed results is made in Table IV. The data is quoted from a forthcoming paper on the thermodynamic properties of hydrogen by the authors. The major error in the above procedure is due to the use of the four point interpolation coefficients. The maximum error in each of the sums is of the order of $2 \times 10^{-7} \sum p_i$. In cases where the $\sum p_i$ and hence this absolute error might seem to become unduly large, the values of the sums themselves are large and the relative error remains within reasonable limits. It might also be pointed out that this error is approximately half the maximum error attained by the hand computing procedure used previously for the direct summation method. The agreement of the results as given in Table IV is quite satisfactory.

The authors wish to express their thanks to Dr. W. J. Eckert, Dr. L. H. Thomas, and other members of the staff of the Watson Scientific Computing Laboratory for their invaluable assistance in getting this program started.

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DISCUSSION

[Discussion of this paper was omitted because of time limitations.]

TABLE IV
COMPARISON OF THERMODYNAMIC FUNCTIONS
COMPUTED BY HAND AND MACHINE METHODS

Temperature	*Function*	*Hand Computed*	*Machine Computed*	*Per Cent Deviation*
1500°K	ΣA^*	18.76256	18.76255	.00005
	ΣB^*	8.86564	8.86569	.0006
	ΣC^*	9.07780	9.07783	.0008
	$E^0 - E_0^0$	7711.5	7711.51	
	C_p^0	7.7103	7.71025	
	S^0	42.702	42.7018	
	$-(F^0 - E_0^0)/T$	35.574	35.5743	
5000°K	ΣA^*	95.79684	95.80611	.0096
	ΣB^*	69.60681	69.62325	.024
	ΣC^*	90.85673	90.87847	.024
	$E^0 - E_0^0$	31516.1	31518.3	.0070
	C_p^0	9.39459	9.39447	.0013
	S^0	53.0818	53.0823	.0013
	$-(F^0 - E_0^0)/T$	44.7921	44.7923	.0004

Calculation of the Equilibrium Composition of Systems of Many Constituents

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U. S. Bureau of Mines

THE COMPOSITION of a system at chemical equilibrium is easily calculated when there is only a single reaction to be considered. In this case, the concentrations of each constituent can be related to a single variable, "the degree of reaction," and the solution of the mass-action equation is straightforward. Difficulties are encountered if this method is extended to a consideration of two simultaneous equilibria, and when the number of such simultaneous equilibria becomes large, the ordinary methods become very laborious.

There is need for a systematic procedure designed to provide a method for writing down the necessary relations in the form most appropriate for numerical computation. When the number of constituents is large, the relations must usually be solved by an iterative procedure. In the course of an extended program of such calculations, it is usually necessary to formulate a number of computational procedures, in order to assure sufficiently rapid convergence. If the calculations are to be carried out by punched card methods, it is desirable that the smallest possible number of arithmetical operations of different kinds be involved in order to minimize the number of different control panels required. In a recent publication,[1] a systematic procedure for calculating the equilibrium composition of a system of many constituents was presented. This method presents a simple rule for formulating the work program of such calculations, with the result that very little time is required for setting up a particular problem. The systematic nature of the computational procedure makes the method well-adapted to punched card methods. The method has been routinely employed in this laboratory in a long series of such calculations. The method is as easily applied to a mixture with a very large number of constituents as to a mixture with a small number of constituents, although the time required to obtain the solution would be greater for the former case than for the latter.

In the publication cited, the method was developed for systems of a very general nature. In the present communication, we restrict application of the method to the calculation of the equilibrium composition of mixtures consisting of a single homogeneous gas phase, and we assume that the gas phase is adequately described by the ideal gas equation of state. By taking advantage of these restrictions, we are able to formulate a computational method, applicable to these particular cases, which is substantially simpler and more systematic than the more general method. A large number of systems of industrial and academic importance is included in this category.

THE COMPONENTS

In a system containing many constituents, it is possible to select certain constituents which are sufficient to describe the composition completely. By this it is meant that if the system is conceived to consist of the selected constituents only, its gross composition (in terms of the amounts of each chemical element present) is completely defined. The constituents thus sufficient to describe the composition are called the components of the system. An analytical criterion has been published[2] for the choice of the components. In terms of this criterion, the conditions for equilibrium may be written in a form which has a high degree of symmetry and is particularly well adapted for formulating a computational method for the calculation of the equilibrium composition.

The number of constituents of any system depends upon the accuracy with which it is desired to describe its composition. The constituents to be considered must be chosen *a priori,* and this choice usually will imply the neglect of certain equilibria that may be expected to exert a negligible effect on the composition of the system at equilibrium.

Consider a closed system containing s different substances, which are assumed to be in chemical equilibrium. The molecular formula of the ith substance may be represented by

$$Y^{(i)} = X^{(1)}_{a_{i1}} \ldots X^{(k)}_{a_{ik}} \ldots X^{(m)}_{a_{im}} \, , \qquad (1)$$

$i = 1, 2, \ldots s,$ where $X^{(k)}$ is the symbol of the kth element, a_{ik} is the subscript (which may be zero) to this symbol in the formula of the ith substance, and m is the total number of elements represented in the system. For every i, the array of subscripts a_{ik}, $k = 1, 2, \ldots, m$, may be said to define a vector

$$y_i = (a_{i1}, \ldots, a_{ik}, \ldots, a_{im}) \quad , \tag{2}$$

which may be called the formula vector of substance i. If the rank of the matrix of the vector elements a_{ik} is c, it follows from a well-known theorem of algebra that there are c linearly independent vectors, and if $c<s$ there are $(s\text{-}c)$ linearly dependent vectors which may be expressed as linear combinations of the independent vectors. It may be assumed that the independent vectors are designated by the values $1, 2, \ldots, c$ of their index. Then the dependent vectors may be expressed as linear combinations of the form

$$\sum_{j=1}^{c} \nu_{ij}\, y_j = y_i \quad , \tag{3}$$

$i = c+1, c+2, \ldots, s$. To equation (3) there correspond $(s\text{-}c)$ conceivable chemical reactions

$$\sum_{j=1}^{c} \nu_{ij}\, Y^{(j)} = Y^{(i)} \quad , \tag{4}$$

resulting in the formation, from the c substances with linearly independent formula vectors, of the $(s\text{-}c)$ substances with linearly dependent formula vectors. It follows that the specification of c substances such that their formula vectors are linearly independent is sufficient for a description of the composition of the system. Therefore, the number of components of the system equals the rank c of the matrix of the subscripts to the symbols of the elements in the formulas of the substances comprising the system. It may be noted that the choice of independent vectors is not, in general, unique and that, in consequence, the choice of c substances as components and the expression of the remaining $(s\text{-}c)$ substances as products of reactions involving only the chosen components is usually not unique.

This discussion has demonstrated the possibility of a choice of components which makes it possible to express each of the dependent constituents as products of reactions involving components only. Our computational procedure is based upon the possibility of writing down for the case of interest the reactions that are expressed by equations (4). In many cases, it is possible to write these reactions immediately by intuition. In some cases, it may be necessary to formulate the reactions of equations (4) by applying to the system under consideration the steps indicated by equations (1) to (3).

The Basic Equations

According to the phase rule of Gibbs, a system defined by c components existing as an homogeneous gas phase has $(c+1)$ degrees of freedom. The thermodynamic state is defined by the specification of two state variables, the temperature and pressure being an appropriate choice. The gross composition is uniquely defined by the specification of $(c-1)$ composition variables, giving the relative amounts of each element available to the system.* We denote the gram-atom fraction of the kth element by Q_k, and the number of moles of the j-component in the hypothetical mixture consisting of components only (the mole fractions of the dependent constituents being zero) by q_j. The conversation of each element requires that

$$\sum_{j=1}^{c} a_{jk}\, q_j = Q_k \quad , \tag{5}$$

$k = 1, 2, \ldots, c-1$, where a_{jk} is the subscript to the symbol of the kth element in the molecular formula of the jth component. It will be convenient to employ the normalization relation,

$$\sum_{j=1}^{c} q_j = 1 \quad . \tag{6}$$

Equations (5) and (6) consist of c independent, nonhomogeneous, linear equations that can be solved for the quantities q_j. The conservation of each element in the reactions (4) for the formation of the dependent constituents from the components can be expressed in the form

$$x_j + \sum_{i=c+1}^{s} \nu_{ij}\, x_i = q_j/n \quad , \tag{7}$$

$j = 1, 2, \ldots, c$, where x_i and x_j are the mole fractions in the equilibrium mixture of the ith dependent constituent and the jth component respectively, ν_{ij} is the coefficient of the formula of the jth component in the equation for the reaction leading to the formation of the ith dependent constituent, and n is the total number of moles of gas in the equilibrium mixture corresponding to the normalized constants q_j. In view of equation (4),

$$\sum_{j=1}^{c} x_j + \sum_{i=c+1}^{s} \nu_i\, x_i = 1/n \quad , \tag{8}$$

*Since the composition of the system at equilibrium is expressed in terms of the mole fraction of each constituent in the equilibrium mixture, the result is independent of the total size of the system, which may be taken to be any convenient value. From the point of view of thermodynamics, the molecular form in which the elements are introduced to the system is a matter of indifference.

where

$$\nu_i = \sum_{j=1}^{c} \nu_{ij} \ .$$

The mole fractions are subject to the identity relation

$$\sum_{j=1}^{c} x_j + \sum_{i=c+1}^{s} x_i = 1 \ .$$

Therefore, equation (8) becomes

$$1/n = 1 + \sum_{i=c+1}^{s} (\nu_i - 1)\, x_i \ , \tag{9}$$

and equations (7) may be written

$$x_j = q_j - \sum_{i=c+1}^{s} [\nu_{ij} - q_j (\nu_i - 1)]\, x_i \ , \tag{10}$$

$j = 1, 2, \ldots, c$.

The conditions for chemical equilibrium in an ideal gas mixture obeying Dalton's law may be written in the form

$$x_i = k_i \prod_{j=1}^{c} x_j^{\nu_{ij}} \ , \tag{11}$$

where

$$k_i = K_i\, p^{\nu_i - 1} \ ,$$

$i = c + 1, \ldots, s$, and where p is the pressure and k_i is the thermodynamic equilibrium constant for the reaction leading to the formation of the ith dependent constituent from the components. The equilibrium constants K_i are independent of the particular system under consideration and are functions of the temperature T only.

The computation of the equilibrium composition requires the simultaneous solution of equations (10) and (11). If $x_i << x$ for all i and j, the solution may be carried out by the simple iteration method.[3] An approximate set of values is chosen for the x_j. (In the absence of any criteria for the choice of the initial set, one may take $x_j = q_j$.) Equations (11) are employed in the computation of corresponding values of the x_i. These in turn are employed with equations (10) for the determination of an improved set of values for the x_j. This iterative process is continued until the difference between successive approximations to the x_j is less than the desired precision of the computation.

The convergence of this simple iteration method is very slow for larger relative values of the x_i, and when the x_i and x_j are of the same order of magnitude, this method may not converge at all for any choice of components. A more powerful computational procedure is provided by the Newton-Raphson method.[4] Equations (10) may be written in the form

$$F_j = q_j - x_j - \sum_{i=c+1}^{s} [\nu_{ij} - q_j (\nu_i - 1)]\, x_i \ , \tag{12}$$

$j = 1, 2, \ldots, c$. We seek the solution of the equations $F_j = 0$, subject to equations (11). If the functions F_j are expanded in Taylor series about an approximate set of values of the variables x_j with neglect of terms involving derivatives of second and higher orders, there results a set of c linear equations which can be compactly represented in the notation of matrices by

$$[A_{jj'}^{(r)}]\,[h_{j'}^{(r)}] = [F_j^{(r)}] \ , \tag{13}$$

where the rth and $(r+1)$th approximations to the composition are related by

$$x_j^{(r+1)} = x_j^{(r)} (1 + h_j^{(r)}) \ , \tag{14}$$

and where the elements of the matrix are given by

$$A_{jj'} = x_j\, \delta_{jj'} + \sum_{i=c+1}^{s} \nu_{ij'}\, [\nu_{ij} - q_j (\nu_i - 1)]\, x_i \ . \tag{15}$$

The superscript r indicates that the designated quantity is to be evaluated with the rth approximation to the composition of the system, and $\delta_{jj'}$ is the Kronecker delta.

Criteria for the choice of components that results in the most rapid convergence of the iteration process can be developed from the remainders to the two-term Taylor series expansions of functions F_j. However, the resulting expressions are too cumbersome for practical utility, and in practice the convergence will be found to be satisfactory if the components are selected so as to minimize the quantities k_i, $i = c + 1, \ldots, s$.

Notes on Computational Procedure

In this present section we will describe in some detail the computational procedure based upon the basic equations developed in the preceding sections. In this laboratory an extensive program of such calculations is being carried out with punched card equipment which includes a Type 602 Calculating Punch. The calculations could be very easily performed utilizing a Type 604 Electronic Calculating Punch and programs similar to those described in another contribution to this Forum.* The procedure for setting up a particular problem will be found to be quite routine in nature. This procedure will involve the following steps:

*It may be noted that these methods have been successfully employed in an extended series of computations performed by the Bureau of Mines on the Electronic Numerical Integrator and Calculator (ENIAC) at the Aberdeen Proving Ground.

1. Select a suitable set of components. This selection can usually be made by intuition. In some cases it may be necessary to follow the formal procedure previously outlined. Subject to the requirement that the components be linearly independent in the sense defined above, it is usually desirable to select those constituents that are most abundant in the equilibrium mixture.

2. Write the chemical equations in the form of equations (4) which express the formation of the dependent constituents from the components. These equations can usually be written by inspection. In some cases it may be necessary to employ the formal methods previously developed.

3. Construct a table of the coefficients of equations (4) ν_{ij}; $j = 1, 2, \ldots, c$; $i = c+1, c+2, \ldots, s$. Tabulate the quantities $(\nu_i - 1)$. If the solution is to be by the Newton-Raphson method, form the products $\nu_{ij}\,\nu_{ij'}$ and $\nu_{ij}\,(\nu_i - 1)$; $j, j' = 1, 2, \ldots, c$; $i = c+1, c+2, \ldots, s$.

4. If the solution is to be by the iteration method, write the explicit form of equations (10) and (11) applicable to the problem under consideration, using the table of coefficients. If the solution is to be by the Newton-Raphson method, write the explicit form of equations (11), (12), and (15) applicable to the problem under consideration, using the table of coefficients.

5. In the usual problem a relatively small number of different gross compositions is considered and computations are performed for a variety of different temperatures and pressures. Under these conditions the values of the stoichiometric constants q_j are most easily obtained from the specifications of the problem by desk calculations based upon equations (5) and (6). If a large number of different mixtures is to be considered, it may be desirable to formulate a routine involving punched card methods for these calculations.

The computational procedure that has been developed appears to be particularly well adapted to the application of punched card methods. The procedure involves several instances where a number of arithmetical operations of the same general kind are performed. In each instance, these operations can be performed with a single control panel that requires only modification of the factor wiring. The notes that follow are based upon experience gained in this laboratory in calculating equilibrium composition and are presented in the hope that they may be useful to other investigators confronted with the same type of problem.

1. Select a first approximate set of values of the mole fractions of the components x_j. In the absence of data by which a more precise set may be estimated, one can adopt the values $x_j = q_j$ for all j.

2. Compute the quantity,

$$\prod_{j=1}^{c} x_j^{\nu_{ij}}.$$

The operations involved in this computation are peculiar to the specific problem at hand. The design of control panels for these steps is routine. The coefficients ν_{ij} are small integers or simple fractions.

3. Complete the calculation of the values of the mole fractions of the dependent constituents x_i corresponding to the approximate values of the x_j by means of equation (11). For this purpose it is useful to sort the cards into groups according to the temperature and pressure and to file each such group behind a master card containing the equilibrium constants k_i, which are employed as group multipliers. It is also useful to punch the answers x_i on trailer cards which contain a suitable identification code and are prepunched with the quantities

ν_{ij}, $\nu_{ij}\,\nu_{ij'}$, $(\nu_i - 1)$, $\nu_{ij}\,(\nu_i - 1)$; $j, j' = 1, 2, \ldots, c$.

A separate set of these trailer cards is employed for each value of i. The appropriate set of trailer cards is inserted by means of the collator prior to the calculation of a particular x_i and withdrawn by selection after the calculation.

4. Calculate the quantities F_j, $j = 1, 2, \ldots, c$. For this purpose we rewrite equation (12) in the form

$$F_j = q_j - x_j - \sum_{i=c+1}^{s} \nu_{ij}\,x_i + q_j \sum_{i=c+1}^{s} (\nu_i - 1)\,x_i\,. \qquad (16)$$

In this calculation we have employed the control panel illustrated in Figure 1. The trailer cards from the previous step are sorted into groups with constant values of the q_j, and each such group is filed behind a master card containing the q_j. The figure assumes that the cards containing the x_i are identified by an X punch in position to be read by the fourth control brush, and the master cards are identified by an X punch in position to be read by the fifth control brush. Each group of the cards containing the x_i is followed by an appropriately identified trailer card that contains the quantities x_j and an X punch which we here assume to be in position to be read by the first control brush. The quantity q_j is read from the master card and held in the summary counter until the next master card is read. The master cards are ejected without punching. For each card containing the x_i, the products $\nu_{ij}\,x_i$ and $(\nu_i - 1)\,x_i$ are formed and accumulated in the left components counter. These cards are ejected without punching. The computation is completed on the trailer card and F_j is punched in this card.

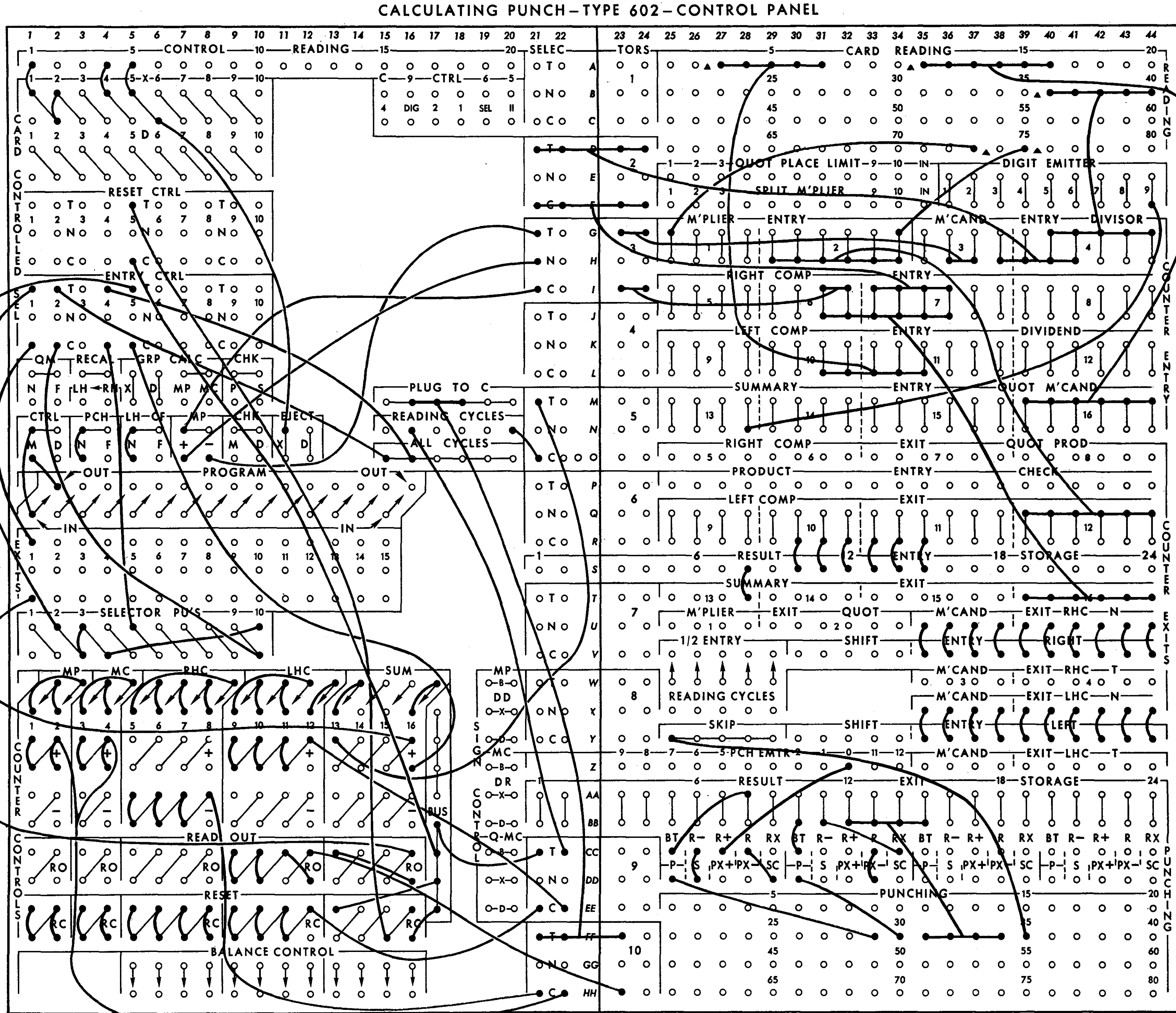

FIGURE 1. TYPE 602 CONTROL PANEL FOR CALCULATING THE COEFFICIENTS OF EQUATION (13)

5. Examine the absolute values of the F_j, and sort out those cases that fulfill a convergence criterion of the form

$$\sum_{j=1}^{c} |F_j| \leqq \epsilon \ , \qquad (17)$$

where ϵ is a preassigned small number measuring the desired precision of the calculation. If the calculation is being made by the iteration method, determine an improved set of values of the x_j according to the relation

$$x_j^{(r+1)} = x_j^{(r)} + F_j \ , \qquad (18)$$

$j = 1, 2, \ldots, c$, where $x_j^{(r)}$ and $x_j^{(r+1)}$ denote the rth and $(r+1)$th approximation to the composition of the system. The computational sequence beginning with step (2) is then repeated. If the calculation is made by the Newton-Raphson method, proceed to step (6).

6. Calculate the coefficients $A_{jj'}$ of equations (13) by means of equations (15) which may be written in the form

$$A_{jj'} = x_j \, \delta_{jj'} + \textstyle\sum_i \nu_{ij} \nu_{ij'} x_i - q_j \sum_i \nu_{ij'} (\nu_i - 1) \, x_i \ , \qquad (19)$$

where $\delta_{jj'}$ is the Kronecker delta which equals one for $j' = j$ and zero for $j' \neq j$. The control panel illustrated in Figure 1 may be employed in these calculations. In these calculations the control panel is modified in the following manner: the quantity q_j is read out from the summary counter to the multiplicand, but not to the right components counter; the brushes which are shown reading the quantities ν_{ij} and $(\nu_i - 1)$ are employed to read $\nu_{ij} \nu_{ij'}$ and $\nu_{ij} (\nu_i - 1)$, respectively. In addition, when $j' \neq j$ the quantity x_j is not read into the left component counter.

7. Solve equations (13), and employ equations (14) to determine an improved set of x_j. The computational procedure beginning with step (2) is then repeated. In solving equations (13), we have employed Cramer's method for cases where the rank of the matrix is less than four. Under these circumstances, we have found it easy to program the evaluation of the necessary determinants, and the method has the advantage that only a single division is required. Since the number of multiplications required by this method increases rapidly with the rank of the matrix, it is desirable to employ more systematic procedures for the reduction of the matrix when the rank is greater than or equal to four. Reference is made to the method presented by Mr. Bell.

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DISCUSSION

[Discussion of this paper was omitted because of time limitations.]

Punched Card Calculating and Printing Methods in the Nautical Almanac Office

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THE IBM EQUIPMENT at the United States Naval Observatory is used primarily to serve the needs of astronomers, navigators, and aviators. We publish annually the *American Ephemeris and Nautical Almanac*, the *American Nautical Almanac*, and the *American Air Almanac*.

The *Ephemeris* supplies data with the highest degree of accuracy. It contains theoretical positions of the sun, moon, planets and stars, with an accuracy of a tenth of a second of arc. Astronomers compare their observations with these data. The *Air Almanac*, on the other hand, is very rough by these standards, giving to the nearest minute of arc positions of the sun, three chosen planets and the moon for every ten minutes of the day. In other words, it is a highly interpolated table of relatively low accuracy to enable an aviator to determine his position rapidly during a flight.

The *Nautical Almanac* stands halfway between the other two, with an accuracy of a tenth of a minute of arc, and values given for every hour. The new form of the *Nautical Almanac*, which will be published beginning with the year 1950, is designed to facilitate use by having all the essential data for three days at one opening of the book. The *Air Almanac* was designed similarly; all the information necessary for the aviator except a few tables is available on one page opening. This includes corrections for the parallax of the moon, tables of moonrise and moonset, and sunrise and sunset, in addition to the values of the Greenwich hour angle (GHA) and declination for various objects. The star tables are not included on the daily page of the *Air Almanac*, because, to this accuracy, a star's position remains practically the same from day to day throughout the year.

All these data for the two almanacs are prepared by IBM machines from the accurate material of the *Ephemeris*. In addition, all the information is arranged on cards so that it may be printed on a special model card-operated typewriter, of which there are only two in existence. The U. S. Naval Observatory has the first of these machines;[1] the other is at IBM World Headquarters in New York City. They are modifications of the standard card-operated typewriter, special as to keyboard, and as to the type of work they can produce.

I am going to devote some attention to the typewriter and attempt to demonstrate the versatility of its output by means of illustrations, but I want to show also how the rest of our IBM equipment fits into the picture, in supplying material for the typewriter.

There are some special problems in setting up the cards, and that is where the other machines come in. For one thing, in a table which is published for a navigator or aviator, accuracy is essential. That means the methods of checking the results must be practically fool-proof. It is necessary that the typewriter prepare the copy from single punched cards. By that I mean each column containing numerical information must have just one punch, because our method of proofreading is to take the printer's proof as it is returned after a photo-offset plate has been made and punch the material again line for line on a new set of cards. Then those cards are compared on the 513 with the cards which had been used to prepare the copy.

That sounds like a painful process when you consider that the *Air Almanac* each year consists of 730 pages of 72 lines, each line consisting of a solid row of figures. It requires a considerable amount of punching to duplicate all that. Yet that is the most accurate, and even turns out to be the fastest, operation. The method is applied also to the other publications, and so for any job we do, the first requirement is that the numerical data occupy single punched columns. The only double punches permitted on the detail cards are code punches, which are not punched in proof.

In addition to the comparison with the original copy cards, a further proof against errors is made by differencing the functions. This is carried out on the 405. Since we have several functions on a card, the method is limited to forming first differences, and summary punching these to get second, and so on. It is a slow process to get sixth differences this way, but that is what is done, for example, in the case of the ephemeris of Mercury.

The work is by no means all proofreading. As an example of computational work I should like to explain the moonrise and moonset tables which are computed every year.

The lunar ephemeris, which gives the position of the moon for every hour of every day, is combined with a permanent table of cards which need be made up only once on the collator so that from the ephemeris are extracted those times of each day at which a moonrise or a moonset is possible at certain selected latitudes. The data from the ephemeris are then reproduced into new cards, and these in turn form the basis for an inverse interpolation to find the exact time of the phenomenon. Values of an hour angle preceding and of an hour angle following the moon rise or set are obtained in the 405 and summary punched. An interpolation on the 601 or 602 gives the accurate time. Corrections are applied to the hour angles for the moon's parallax and for its motion in declination before the interpolation is made. Except for the original hand punching and checking of the various permanent tables, everything is done by the IBM machines. The only hand work is a spot check every year on a few of the computed values. Other checks include differencing day by day for each latitude, and differencing values for the corresponding day with respect to latitude. The latter is a most powerful check, for if one adds the moonrise time to the moonset time for each latitude for a given day, the sum is practically a constant for all latitudes. This check is applied also to the sunrise and twilight tables of the *Nautical Almanac*.

The only fault to be found with this whole procedure is that a great deal of hand manipulation of large decks of cards is required because there is no way of moving cards from one machine to another except by hand. Considerable care is necessary to prevent disarrangement of the cards. At some stages in the process, any such disarrangement could go undetected long enough to cause serious damage. The magnitude of the task is obvious when you consider that we compute moonrise and moonset for thirty-four selected latitudes from $-60°$ to $+73°$ for every day of the year.

Another major task is, of course, the computation of the Greenwich hour angles and declinations in the *Nautical Almanac* and *Air Almanac*. These are subtabulated from the daily values in the *American Ephemeris*, by computing the hourly differences, and by progressive totaling these differences with appropriate starting values, thereby building up the required functions. The data for the *Nautical Almanac* are further subtabulated to give the ten minute interval of the *Air Almanac*.

The original installation of the IBM machines at the Observatory was for the purpose of preparing the *Air Almanac* for publication. Since that time, more and more work has been transferred from the computers' desks to punched cards. Among such jobs I might mention the computation of occultation elements: the time when the moon will occult a star; apparent star places; precession and reduction to mean places of stars; heliocentric coordinates of the major planets; research in the theory of the motions of the major planets and their satellites. Much of this work is annual, but some of it has been done just once.

The methods we use do not equal the complexity demonstrated in the solutions to some of the problems which have been presented earlier. As I have mentioned, we form differences of the first order and summary punch them to get second differences. We subtabulate wherever possible, rather than interpolate, because subtabulation goes faster and more automatically and, once it is set up, several functions can be done simultaneously. Checking is made easier by this method, also. Our intervals are usually uniform, and we have no problems involving complex quantities or matrices.

We make one principal demand, and that is the utmost in accuracy to a large number of decimal places. In the integration of planetary orbits, ten decimal place accuracy is common. In our almanac work, extra decimals are needed to prevent accumulated rounding errors. Both the large capacity counters and the accuracy are supplied in a satisfactory way by the Type 405 Accounting Machine and the Type 602 Calculating Punch. One handicap is the limited punching capacity of the Type 602.

Now I would like to tell you a little about the card-operated typewriter. Perhaps the best way is to refer to the illustrations; then a few comments will give you a good idea of its capabilities. Figure 1 is a page of the *Air Almanac*. It represents the time from noon to midnight of the same day for every ten minutes of time, giving for those times the GHA's and declinations of the sun, moon, and three planets, and in addition, tables of sunrise and moonrise. The summary punching which produces the GHA's also produces the cards which will go through the typewriter; the declinations and miscellaneous tables are reproduced into these cards in their proper line relationship. The typewriter prints the numbers from these detail cards, and the control of the spacing from column to column is taken care of by a master card which is read over again for each new line.

GCT	☉ SUN GHA	SUN Dec.	♈ GHA	VENUS −4.0 GHA	VENUS Dec.	MARS 1.5 GHA	MARS Dec.	JUPITER −1.9 GHA	JUPITER Dec.	◑ MOON GHA	MOON Dec.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
12 00	359 34	N10 22	154 39	46 07	N18 55	311 48	S 9 44	256 18	S22 46	109 24	N18 08
10	2 04		157 09	48 37		314 18		258 48		111 49	10
20	4 34		159 39	51 07		316 48		261 18		114 14	12
30	7 04	· ·	162 10	53 37	· ·	319 18	· ·	263 49	· ·	116 40	· 14
40	9 34		164 40	56 07		321 49		266 19		119 05	16
50	12 04		167 11	58 37		324 19		268 49		121 30	18
13 00	14 34	N10 21	169 41	61 07	N18 55	326 49	S 9 45	271 20	S22 46	123 55	N18 19
10	17 04		172 11	63 37		329 19		273 50		126 21	21
20	19 34		174 42	66 07		331 49		276 21		128 46	23
30	22 05	· ·	177 12	68 37	· ·	334 19	· ·	278 51	· ·	131 11	· 25
40	24 35		179 43	71 07		336 49		281 21		133 36	27
50	27 05		182 13	73 37		339 20		283 52		136 01	29
14 00	29 35	N10 20	184 43	76 07	N18 55	341 50	S 9 46	286 22	S22 46	138 27	N18 31
10	32 05		187 14	78 37		344 20		288 53		140 52	33
20	34 35		189 44	81 07		346 50		291 23		143 17	34
30	37 05	· ·	192 15	83 37	· ·	349 20	· ·	293 53	· ·	145 42	· 36
40	39 35		194 45	86 07		351 50		296 24		148 08	38
50	42 05		197 15	88 37		354 21		298 54		150 33	40
15 00	44 35	N10 19	199 46	91 07	N18 55	356 51	S 9 46	301 25	S22 46	152 58	N18 42
10	47 05		202 16	93 37		359 21		303 55		155 23	44
20	49 35		204 47	96 07		1 51		306 25		157 48	46
30	52 05	· ·	207 17	98 37	· ·	4 21	· ·	308 56	· ·	160 14	· 48
40	54 35		209 48	101 07		6 51		311 26		162 39	49
50	57 05		212 18	103 37		9 22		313 57		165 04	51
16 00	59 35	N10 18	214 48	106 07	N18 55	11 52	S 9 47	316 27	S22 46	167 29	N18 53
10	62 05		217 19	108 37		14 22		318 57		169 54	55
20	64 35		219 49	111 07		16 52		321 28		172 20	57
30	67 05	· ·	222 20	113 37	· ·	19 22	· ·	323 58	· ·	174 45	18 59
40	69 35		224 50	116 07		21 52		326 29		177 10	19 01
50	72 05		227 20	118 37		24 22		328 59		179 35	02
17 00	74 35	N10 17	229 51	121 07	N18 54	26 53	S 9 48	331 29	S22 46	182 00	N19 04
10	77 05		232 21	123 37		29 23		334 00		184 26	06
20	79 35		234 52	126 07		31 53		336 30		186 51	08
30	82 05	· ·	237 22	128 37	· ·	34 23	· ·	339 01	· ·	189 16	· 10
40	84 35		239 52	131 07		36 53		341 31		191 41	12
50	87 05		242 23	133 37		39 23		344 01		194 06	13
18 00	89 35	N10 16	244 53	136 07	N18 54	41 54	S 9 48	346 32	S22 46	196 31	N19 15
10	92 05		247 24	138 37		44 24		349 02		198 57	17
20	94 35		249 54	141 07		46 54		351 33		201 22	19
30	97 05	· ·	252 25	143 37	· ·	49 24	· ·	354 03	· ·	203 47	· 21
40	99 35		254 55	146 07		51 54		356 33		206 12	23
50	102 05		257 25	148 37		54 24		359 04		208 37	24
19 00	104 35	N10 15	259 56	151 07	N18 54	56 55	S 9 49	1 34	S22 46	211 02	N19 26
10	107 05		262 26	153 37		59 25		4 05		213 27	28
20	109 36		264 57	156 08		61 55		6 35		215 53	30
30	112 06	· ·	267 27	158 38	· ·	64 25	· ·	9 05	· ·	218 18	· 32
40	114 36		269 57	161 08		66 55		11 36		220 43	33
50	117 06		272 28	163 38		69 25		14 06		223 08	35
20 00	119 36	N10 15	274 58	166 08	N18 54	71 56	S 9 49	16 37	S22 46	225 33	N19 37
10	122 06		277 29	168 38		74 26		19 07		227 58	39
20	124 36		279 59	171 08		76 56		21 37		230 23	41
30	127 06	· ·	282 29	173 38	· ·	79 26	· ·	24 08	· ·	232 48	· 42
40	129 36		285 00	176 08		81 56		26 38		235 14	44
50	132 06		287 30	178 38		84 26		29 09		237 39	46
21 00	134 36	N10 14	290 01	181 08	N18 54	86 56	S 9 50	31 39	S22 46	240 04	N19 48
10	137 06		292 31	183 38		89 27		34 09		242 29	50
20	139 36		295 02	186 08		91 57		36 40		244 54	51
30	142 06	· ·	297 32	188 38	· ·	94 27	· ·	39 10	· ·	247 19	· 53
40	144 36		300 02	191 08		96 57		41 41		249 44	55
50	147 06		302 33	193 38		99 27		44 11		252 09	57
22 00	149 36	N10 13	305 03	196 08	N18 54	101 57	S 9 51	46 41	S22 46	254 34	N19 59
10	152 06		307 34	198 38		104 28		49 12		256 59	20 00
20	154 36		310 04	201 08		106 58		51 42		259 25	02
30	157 06	· ·	312 34	203 38	· ·	109 28	· ·	54 13	· ·	261 50	· 04
40	159 36		315 05	206 08		111 58		56 43		264 15	06
50	162 06		317 35	208 38		114 28		59 13		266 40	07
23 00	164 36	N10 12	320 06	211 08	N18 54	116 58	S 9 51	61 44	S22 46	269 05	N20 09
10	167 06		322 36	213 38		119 29		64 14		271 30	11
20	169 36		325 06	216 08		121 59		66 44		273 55	13
30	172 06	· ·	327 37	218 38	· ·	124 29	· ·	69 15	· ·	276 20	· 14
40	174 36		330 07	221 08		126 59		71 45		278 45	16
50	177 06		332 38	223 38		129 29		74 16		281 10	18
24 00	179 36	N10 11	335 08	226 08	N18 54	131 59	S 9 52	76 46	S22 46	283 35	N20 20

Lat.	Sunrise	Twlt.	Moonrise	Diff.
N °	h m	m	h m	m
70	3 48	88	18 04	*
68	4 02	73	18 55	*
66	14	63	19 27	*
64	24	57	19 52	*
62	32	51	20 11	*
60	39	46	27	06
58	46	42	40	12
56	51	39	20 52	16
54	4 57	38	21 02	20
52	5 01	36	11	23
50	05	34	19	26
45	14	30	37	31
40	21	28	21 51	35
35	28	26	22 03	38
30	33	24	14	40
20	43	23	33	44
10	51	22	22 49	48
0	5 58	21	23 05	51
10	6 06	22	20	55
20	14	23	36	59
30	23	25	23 55	63
35	28	26	24 07	65
40	33	28	20	68
45	40	31	35	72
50	48	34	24 54	78
52	52	36	25 03	80
54	6 56	38	25 13	84
56	7 00	40	0 00	85
58	05	42	09	89
60	7 11	44	0 20	94
S				

Lat.	Sunset	Twlt.	Moonset	Diff.
N °	h m	m	h m	m
70	20 12	86	15 21	*
68	19 58	72	14 31	*
66	47	62	14 00	144
64	37	56	13 37	118
62	29	51	18	105
60	23	47	13 04	96
58	16	44	12 51	90
56	11	41	40	86
54	06	38	30	82
52	19 01	36	22	79
50	18 57	34	12 14	77
45	49	31	11 58	71
40	41	28	45	67
35	35	26	34	64
30	30	25	24	61
20	20	23	11 07	57
10	12	22	10 53	52
0	18 05	21	39	49
10	17 58	21	26	45
20	50	23	10 12	41
30	41	24	9 56	36
35	36	26	46	34
40	31	28	35	31
45	24	30	23	26
50	16	32	08	21
52	13	34	9 00	20
54	09	36	8 53	16
56	04	39	44	13
58	17 00	41	34	09
60	16 54	43	8 23	03
S				

FIGURE 1

Day	JULY Rise	t	y	Set	AUGUST Rise	t	y	Set	SEPTEMBER Rise	t	y	Set	OCTOBER Rise	t	y	Set	NOVEMBER Rise	t	y	Set	DECEMBER Rise	t	y	Set
1	□	••	••	□	0213	+20	−5	2155	0425	+5	−5	1933	0612	−2	−4	1726	0810	−9	−4	1516	1035	−64	−5	1302
2	□	••	••	□	0219	+19	−5	2150	0429	+5	−4	1928	0616	−2	−5	1722	0814	−10	−4	1512	1041	••	−6	1257
3	□	••	••	□	0224	+17	−5	2145	0432	+5	−4	1924	0619	−1	−4	1717	0818	−10	−5	1508	1048	••	−5	1251
4	□	••	••	□	0229	+17	−5	2140	0436	+4	−5	1920	0623	−2	−4	1713	0823	−10	−4	1503	1054	••	−7	1246
5	□	••	••	□	0233	+16	−5	2135	0440	+5	−4	1915	0626	−2	−4	1709	0827	−11	−4	1459	1101	••	−7	1239
6	□	••	••	□	0238	+15	−5	2130	0443	+4	−4	1911	0630	−2	−4	1705	0831	−11	−4	1455	1109	••	−7	1232
7	□	••	••	□	0243	+15	−5	2125	0447	+4	−4	1907	0633	−3	−4	1701	0836	−12	−4	1451	1118	••	−10	1225
8	□	••	••	□	0247	+14	−5	2120	0451	+3	−5	1903	0637	−3	−5	1657	0840	−13	−5	1447	1129	••	••	1215
9	□	••	••	□	0252	+14	−4	2115	0454	+4	−4	1858	0641	−3	−4	1652	0844	−12	−4	1442	■	••	••	■
10	□	••	••	□	0257	+13	−5	2111	0458	+3	−4	1854	0644	−3	−4	1648	0849	−13	−4	1438	■	••	••	■
11	□	••	••	□	0301	+12	−5	2106	0501	+3	−4	1850	0648	−3	−4	1644	0853	−14	−5	1434	■	••	••	■
12	□	••	••	□	0305	+12	−4	2101	0505	+2	−5	1846	0652	−4	−4	1640	0858	−14	−4	1429	■	••	••	■
13	□	••	••	□	0310	+11	−5	2057	0508	+3	−4	1841	0655	−4	−4	1636	0903	−15	−4	1425	■	••	••	■
14	□	••	••	□	0314	+11	−4	2052	0512	+2	−4	1837	0659	−5	−5	1632	0907	−16	−5	1421	■	••	••	■
15	□	••	••	□	0318	+10	−5	2048	0515	+2	−4	1833	0703	−4	−4	1627	0912	−16	−4	1416	■	••	••	■
16	□	••	••	□	0322	+10	−4	2043	0519	+1	−4	1829	0707	−4	−4	1623	0917	−17	−4	1412	■	••	••	■
17	□	••	−17	2348	0326	+ 9	−5	2039	0522	+1	−5	1825	0710	−5	−4	1619	0921	−18	−5	1408	■	••	••	■
18	0025	••	−11	2331	0331	+ 9	−4	2034	0526	+2	−4	1820	0714	−5	−4	1615	0926	−18	−4	1403	■	••	••	■
19	0042	••	−10	2320	0335	+ 8	−5	2030	0529	+1	−4	1816	0718	−6	−5	1611	0931	−20	−5	1359	■	••	••	■
20	0054	••	− 8	2310	0339	+ 9	−4	2025	0533	+1	−4	1812	0722	−5	−4	1606	0936	−20	−4	1354	■	••	••	■
21	0103	••	− 8	2302	0343	+ 8	−5	2021	0537	0	−4	1808	0726	−6	−4	1602	0941	−22	−5	1350	■	••	••	■
22	0112	••	− 7	2254	0347	+ 8	−4	2016	0540	0	−5	1804	0729	−6	−4	1558	0946	−23	−4	1345	■	••	••	■
23	0119	+46	− 6	2247	0351	+ 7	−5	2012	0544	+1	−4	1759	0733	−6	−4	1554	0951	−25	−5	1341	■	••	••	■
24	0126	+38	− 7	2241	0355	+ 8	−4	2007	0547	0	−4	1755	0737	−7	−4	1550	0956	−26	−4	1336	■	••	••	■
25	0133	+34	− 6	2234	0359	+ 7	−4	2003	0551	0	−4	1751	0741	−7	−5	1546	1002	−29	−5	1332	■	••	••	■
26	0140	+31	− 6	2228	0402	+ 6	−5	1959	0554	−1	−4	1747	0745	−7	−4	1541	1007	−30	−5	1327	■	••	••	■
27	0146	+28	− 6	2222	0406	+ 7	−4	1954	0558	−1	−5	1743	0749	−7	−4	1537	1012	−33	−5	1322	■	••	••	■
28	0152	+26	− 5	2216	0410	+ 6	−4	1950	0601	0	−4	1738	0753	−8	−4	1533	1018	−36	−5	1317	■	••	••	■
29	0157	+24	− 6	2211	0414	+ 5	−5	1946	0605	−1	−4	1734	0758	−8	−4	1529	1024	−41	−5	1312	■	••	••	■
30	0203	+23	− 5	2205	0418	+ 6	−4	1941	0608	−1	−4	1730	0802	−9	−5	1525	1029	−48	−5	1307	■	••	••	■
31	0208	+21	− 5	2200	0421	+ 5	−4	1937	•••	•	•	•••	0806	−8	−4	1520	•••	••	•	•••	■	••	••	■

Day	JULY Twilight Civ.	Nt.	Ast.	Daylight	AUGUST Twilight Civ.	Nt.	Ast.	Daylight	SEPTEMBER Twilight Civ.	Nt.	Ast.	Daylight	OCTOBER Twilight Civ.	Nt.	Ast.	Daylight	NOVEMBER Twilight Civ.	Nt.	Ast.	Daylight	DECEMBER Twilight Civ.	Nt.	Ast.	Daylight
1	□	□	□	□	////	////	////	1942	105	251	////	1508	055	201	316	1114	105	212	316	0706	152	314	423	0227
3	□	□	□	□	////	////	////	1921	104	242	////	1452	055	201	313	1058	106	214	318	0650	159	322	432	0203
5	□	□	□	□	////	////	////	1902	102	235	////	1435	055	200	311	1043	108	216	321	0632	208	332	442	0138
7	□	□	□	□	////	////	////	1842	101	229	////	1420	056	200	310	1028	110	219	323	0615	219	344	455	0107
9	□	□	□	□	211	////	////	1823	100	224	////	1404	056	200	309	1011	112	222	326	0558	249	416	526	■
11	□	□	□	□	152	////	////	1805	059	220	////	1349	056	200	308	0956	114	225	330	0541	246	413	524	■
13	□	□	□	□	141	////	////	1747	058	216	431	1333	057	201	308	0941	116	228	333	0522	244	412	523	■
15	□	□	□	□	133	////	////	1730	058	213	408	1318	057	201	307	0924	119	231	337	0504	242	410	522	■
17	////	////	////	2348	127	////	////	1713	057	211	355	1303	058	202	307	0909	121	235	341	0447	240	409	521	■
19	////	////	////	2238	123	////	////	1655	057	209	345	1247	058	203	308	0853	125	240	346	0428	239	409	520	■
21	////	////	////	2159	119	////	////	1638	056	207	337	1231	059	204	308	0836	128	244	351	0409	239	408	520	■
23	////	////	////	2128	115	////	////	1621	056	205	331	1215	100	205	309	0821	132	249	356	0350	239	408	520	■
25	////	////	////	2101	112	////	////	1604	055	204	326	1200	101	206	310	0805	136	254	402	0330	240	409	520	■
27	////	////	////	2036	110	342	////	1548	055	203	322	1145	102	207	312	0748	141	300	408	0310	241	410	521	■
29	////	////	////	2014	108	312	////	1532	055	202	319	1129	103	209	313	0731	146	307	415	0248	243	411	522	■
31	////	////	////	1952	106	257	////	1516	•••	•••	•••	•••	104	211	315	0714	•••	•••	•••	•••	245	413	524	■

To obtain the values for other than integral degrees of latitude, see pages 14–15.

With the dates as given, all values are for northern latitudes. For *southern latitudes*, see page 15.

FIGURE 2

Figure 2 is a sample of the sunrise and sunset table which was computed by Dr. Herget. It was published by the Naval Observatory in 1945. The open squares represent continuous sunshine, the black squares continuous night, and the four diagonal strokes continuous twilight. Choosing the small type in printing is done by the master card; the numbers punched on the detail cards have no special code punches. What this means is that a variety of type styles and formats is available for any set of detail cards.

In this case, a complication arose from the fact that the three blocks July, August, and September, on the left-hand side of the page, were on one set of cards, and the blocks October, November, and December were on another set. The combination totaled more than eighty columns, so that it wasn't possible to put both these fields on one card. The solution was simple but tedious. The first quarter of the page was run down to the first heavy line, and then the page was rolled back. With a new master card in the reading unit, the second set of cards was read. The typewriter skipped over the previously printed matter and placed the data in their proper place on the page. When the heavy line was reached a second time, the whole process was repeated for the twilights printed below the line. Thus, each page was done in four parts without removing it from the typewriter. The present day solution would be simpler still: the typewriter is able to read a card and print information from it, then eject the card without returning the carriage and continue with the next card on the same line; finally it ejects that card and returns the carriage for a new line.

The planet page (Figure 3) of the 1950 *Nautical Almanac* is radically different from the star page (Figure 4), yet both are done with the same control panel with six changes of wiring. Six jackplugs are removed to do the planet page, and inserted to do the other. Of course, the master cards are radically different. I am often tempted to think of a super control panel for this machine, with which any printing job could be done merely by punching up the proper master card. One could wire a permanent control panel and with a master card to suit each job do perhaps 95 per cent of all the work demanded. There is a practical limit, of course, which comes from the finite size of the control panel and the number of available selectors. Beyond this point back circuits are bound to occur.

The interpolation table of the *Nautical Almanac* (Figure 5) was constructed on the accounting machine from blank cards by using suitable rolling counters. These pages give, for the objects on the planet and star pages, a proportional parts table which enables the navigator to interpolate without multiplying. We do his multiplying for him, making it simple for the navigator to use the data to find his position.

Figure 6 is a good sample of the versatility of the typewriter. The apparent positions of selected pairs of stars were computed on the IBM machines for the International Latitude Service. The volume was to have very small circulation. It was to consist of perhaps a hundred copies, and we were not prepared to print it. The obvious thing to do was to mimeograph it, and that is what was done. The typewriter did all the stenciling except the heading lines. The remainder, from the star numbers through the symbols and the data, was read from cards and automatically printed. The mimeograph stencils were extremely uniform, due to the electric typewriter action.

The extra upspaces between lines are produced automatically at the proper intervals by code punches in the detail cards. No blank cards are needed. This saves time in printing as well as in arranging the detail cards. Such code punches can be used to suppress unwanted printing or force desired numbers on certain lines. Thus, it is not necessary to make sure that the detail card is blank where no printing is desired.

The *Minor Planet Ephemerides* (Figure 7) are a masterpiece of typing. They illustrate most of the capabilities of the typewriter on one page. The only printing not done by the card-operated machine was the planet names. They were typed in by hand on an IBM proportional spacing typewriter.

Figure 8 is another portion of that table. It gives the elements of the minor planets. In the fourth column Roman numerals denote the month. These were not only printed but spaced from the detail card. The master card couldn't predict ahead of time which of the numerals was going to appear; therefore the detail card had to take over the spacing. That is the closest approach to proportional spacing that the machine has. The typewriter itself contains no direct connection between the key pressed and the space obtained; the spacing is controlled from the reading unit. It supplies as much as eighteen unit spaces, or 9/16 of an inch, from one card column. In general, the amount of space allotted to a character is determined beforehand from the reading unit and the master card, except, of course, in such special cases as the Roman numerals.

It would be appropriate to mention here the problem of point plotting, as it has already arisen in some earlier discussions. Choose one coordinate as the upspace of the platen, and the other as a motion of the carriage along the line. It is possible to arrange a deck of cards which will determine how far the carriage moves before a symbol is printed. If ten inches is the width of the page, then, with 320 unit spaces, the accuracy of the plot will be 0.3 per cent of full scale. Several passages of the same page

GCT	♈	VENUS −3.3		MARS 1.1		JUPITER −2.4		SATURN 1.3		MOON			
	GHA	GHA	Dec.	GHA	Dec.	GHA	Dec.	GHA	Dec.	GHA	code	Dec.	code
h	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′		° ′	
0	342 32.7	197 20.3	+14 56.3	120 02.0	−17 20.8	8 41.4	−12 13.3	169 53.3	+ 5 16.6	280 55.0	114	+25 47.4	+ 67
1	357 35.2	212 19.7	55.4	135 02.8	21.3	23 44.2	13.4	184 55.5	16.4	295 25.4	114	25 54.1	+ 65
2	12 37.6	227 19.1	54.4	150 03.6	21.8	38 47.0	13.5	199 57.7	16.3	309 55.8	113	26 00.6	+ 64
3	27 40.1	242 18.6	·· 53.5	165 04.5	·· 22.3	53 49.8	·· 13.6	214 59.9	·· 16.2	324 26.1	112	·26 07.0	+ 63
4	42 42.6	257 18.0	52.6	180 05.3	22.8	68 52.5	13.8	230 02.1	16.1	338 56.3	111	26 13.3	+ 61
5	57 45.0	272 17.4	51.6	195 06.1	23.3	83 55.3	13.9	245 04.2	16.0	353 26.4	111	26 19.4	+ 61
6	72 47.5	287 16.9	+14 50.7	210 07.0	−17 23.8	98 58.1	−12 14.0	260 06.4	+ 5 15.8	7 56.5	110	+26 25.5	+ 59
7	87 49.9	302 16.3	49.7	225 07.8	24.3	114 00.8	14.1	275 08.6	15.7	22 26.5	110	26 31.4	+ 58
8	102 52.4	317 15.7	48.8	240 08.6	24.8	129 03.6	14.2	290 10.8	15.6	36 56.5	109	26 37.2	+ 57
9	117 54.9	332 15.2	·· 47.9	255 09.5	·· 25.3	144 06.4	·· 14.3	305 13.0	·· 15.5	51 26.4	108	·26 42.9	+ 56
10	132 57.3	347 14.6	46.9	270 10.3	25.8	159 09.1	14.4	320 15.1	15.3	65 56.2	108	26 48.5	+ 54
11	147 59.8	2 14.0	46.0	285 11.1	26.3	174 11.9	14.6	335 17.3	15.2	80 26.0	107	26 53.9	+ 53
12	163 02.3	17 13.5	+14 45.1	300 12.0	−17 26.8	189 14.7	−12 14.7	350 19.5	+ 5 15.1	94 55.7	107	+26 59.2	+ 52
13	178 04.7	32 12.9	44.1	315 12.8	27.3	204 17.4	14.8	5 21.7	15.0	109 25.4	106	27 04.4	+ 51
14	193 07.2	47 12.3	43.2	330 13.6	27.8	219 20.2	14.9	20 23.8	14.9	123 55.0	105	27 09.5	+ 49
15	208 09.7	62 11.8	·· 42.2	345 14.5	·· 28.3	234 23.0	·· 15.0	35 26.0	·· 14.7	138 24.5	105	·27 14.4	+ 48
16	223 12.1	77 11.2	41.3	0 15.3	28.8	249 25.8	15.1	50 28.2	14.6	152 54.0	104	27 19.2	+ 47
17	238 14.6	92 10.6	40.3	15 16.1	29.3	264 28.5	15.3	65 30.4	14.5	167 23.4	103	27 23.9	+ 45
18	253 17.1	107 10.1	+14 39.4	30 16.9	−17 29.8	279 31.3	−12 15.4	80 32.6	+ 5 14.4	181 52.7	103	+27 28.4	+ 44
19	268 19.5	122 09.5	38.5	45 17.8	30.3	294 34.1	15.5	95 34.7	14.3	196 22.0	103	27 32.8	+ 43
20	283 22.0	137 09.0	37.5	60 18.6	30.8	309 36.8	15.6	110 36.9	14.1	210 51.3	102	27 37.1	+ 42
21	298 24.4	152 08.4	·· 36.6	75 19.4	·· 31.3	324 39.6	·· 15.7	125 39.1	·· 14.0	225 20.5	101	·27 41.3	+ 40
22	313 26.9	167 07.8	35.6	90 20.2	31.8	339 42.4	15.8	140 41.3	13.9	239 49.6	101	27 45.3	+ 39
23	328 29.4	182 07.3	34.7	105 21.1	32.3	354 45.1	15.9	155 43.5	13.8	254 18.7	100	27 49.2	+ 38
0	343 31.8	197 06.7	+14 33.7	120 21.9	−17 32.8	9 47.9	−12 16.1	170 45.6	+ 5 13.6	268 47.7	100	+27 53.0	+ 36
1	358 34.3	212 06.1	32.8	135 22.7	33.3	24 50.7	16.2	185 47.8	13.5	283 16.7	99	27 56.6	+ 35
2	13 36.8	227 05.6	31.8	150 23.5	33.8	39 53.4	16.3	200 50.0	13.4	297 45.6	99	28 00.1	+ 34
3	28 39.2	242 05.0	·· 30.9	165 24.4	·· 34.3	54 56.2	·· 16.4	215 52.2	·· 13.3	312 14.5	98	·28 03.5	+ 32
4	43 41.7	257 04.5	29.9	180 25.2	34.8	69 59.0	16.5	230 54.3	13.2	326 43.3	98	28 06.7	+ 31
5	58 44.2	272 03.9	29.0	195 26.0	35.3	85 01.7	16.6	245 56.5	13.0	341 12.1	97	28 09.8	+ 30
6	73 46.6	287 03.4	+14 28.0	210 26.8	−17 35.8	100 04.5	−12 16.7	260 58.7	+ 5 12.9	355 40.8	97	+28 12.8	+ 28
7	88 49.1	302 02.8	27.0	225 27.7	36.3	115 07.3	16.9	276 00.9	12.8	10 09.5	96	28 15.6	+ 27
8	103 51.6	317 02.2	26.1	240 28.5	36.8	130 10.0	17.0	291 03.1	12.7	24 38.1	96	28 18.3	+ 25
9	118 54.0	332 01.7	·· 25.1	255 29.3	·· 37.3	145 12.8	·· 17.1	306 05.2	·· 12.5	39 06.7	96	·28 20.8	+ 25
10	133 56.5	347 01.1	24.2	270 30.1	37.8	160 15.6	17.2	321 07.4	12.4	53 35.3	95	28 23.3	+ 22
11	148 58.9	2 00.6	23.2	285 30.9	38.3	175 18.3	17.3	336 09.6	12.3	68 03.8	94	28 25.5	+ 22
12	164 01.4	17 00.0	+14 22.3	300 31.8	−17 38.8	190 21.1	−12 17.4	351 11.8	+ 5 12.2	82 32.2	94	+28 27.7	+ 19
13	179 03.9	31 59.4	21.3	315 32.6	39.3	205 23.9	17.5	6 14.0	12.1	97 00.6	94	28 29.6	+ 19
14	194 06.3	46 58.9	20.4	330 33.4	39.8	220 26.6	17.7	21 16.1	11.9	111 29.0	93	28 31.5	+ 17
15	209 08.8	61 58.3	·· 19.4	345 34.2	·· 40.3	235 29.4	·· 17.8	36 18.3	·· 11.8	125 57.3	93	·28 33.2	+ 16
16	224 11.3	76 57.8	18.4	0 35.1	40.8	250 32.2	17.9	51 20.5	11.7	140 25.6	93	28 34.8	+ 14
17	239 13.7	91 57.2	17.5	15 35.9	41.3	265 34.9	18.0	66 22.7	11.6	154 53.9	92	28 36.2	+ 13
18	254 16.2	106 56.7	+14 16.5	30 36.7	−17 41.8	280 37.7	−12 18.1	81 24.8	+ 5 11.5	169 22.1	92	+28 37.5	+ 11
19	269 18.7	121 56.1	15.5	45 37.5	42.3	295 40.5	18.2	96 27.0	11.3	183 50.3	91	28 38.6	+ 10
20	284 21.1	136 55.6	14.6	60 38.3	42.8	310 43.2	18.3	111 29.2	11.2	198 18.4	92	28 39.6	+ 9
21	299 23.6	151 55.0	·· 13.6	75 39.1	·· 43.3	325 46.0	·· 18.5	126 31.4	·· 11.1	212 46.6	90	·28 40.5	+ 7
22	314 26.0	166 54.5	12.7	90 40.0	43.8	340 48.8	18.6	141 33.6	11.0	227 14.6	91	28 41.2	+ 6
23	329 28.5	181 53.9	11.7	105 40.8	44.3	355 51.5	18.7	156 35.7	10.8	241 42.7	90	28 41.8	+ 4
0	344 31.0	196 53.4	+14 10.7	120 41.6	−17 44.8	10 54.3	−12 18.8	171 37.9	+ 5 10.7	256 10.7	90	+28 42.2	+ 3
1	359 33.4	211 52.8	09.8	135 42.4	45.2	25 57.1	18.9	186 40.1	10.6	270 38.7	90	28 42.5	+ 1
2	14 35.9	226 52.2	08.8	150 43.2	45.7	40 59.8	19.0	201 42.3	10.5	285 06.7	89	28 42.6	0
3	29 38.4	241 51.7	·· 07.8	165 44.0	·· 46.2	56 02.6	·· 19.1	216 44.4	·· 10.4	299 34.6	90	·28 42.6	− 2
4	44 40.8	256 51.1	06.9	180 44.9	46.7	71 05.4	19.2	231 46.6	10.2	314 02.6	88	28 42.4	− 3
5	59 43.3	271 50.6	05.9	195 45.7	47.2	86 08.1	19.4	246 48.8	10.1	328 30.4	89	28 42.1	− 5
6	74 45.8	286 50.0	+14 04.9	210 46.5	−17 47.7	101 10.9	−12 19.5	261 51.0	+ 5 10.0	342 58.3	89	+28 41.6	− 6
7	89 48.2	301 49.5	03.9	225 47.3	48.2	116 13.6	19.6	276 53.2	09.9	357 26.2	88	28 41.0	− 7
8	104 50.7	316 48.9	03.0	240 48.1	48.7	131 16.4	19.7	291 55.3	09.7	11 54.0	88	28 40.3	− 9
9	119 53.2	331 48.4	·· 02.0	255 48.9	·· 49.2	146 19.2	·· 19.8	306 57.5	·· 09.6	26 21.8	88	·28 39.4	− 11
10	134 55.6	346 47.9	01.0	270 49.7	49.7	161 21.9	19.9	321 59.7	09.5	40 49.6	88	28 38.3	− 12
11	149 58.1	1 47.3	14 00.0	285 50.6	50.2	176 24.7	20.0	337 01.9	09.4	55 17.4	88	28 37.1	− 13
12	165 00.5	16 46.8	+13 59.1	300 51.4	−17 50.7	191 27.5	−12 20.2	352 04.1	+ 5 09.3	69 45.2	87	+28 35.8	− 15
13	180 03.0	31 46.2	58.1	315 52.2	51.2	206 30.2	20.3	7 06.2	09.1	84 12.9	88	28 34.3	− 16
14	195 05.5	46 45.7	57.1	330 53.0	51.7	221 33.0	20.4	22 08.4	09.0	98 40.7	87	28 32.7	− 18
15	210 07.9	61 45.1	·· 56.1	345 53.8	·· 52.2	236 35.8	·· 20.5	37 10.6	·· 08.9	113 08.4	87	·28 30.9	− 20
16	225 10.4	76 44.6	55.2	0 54.6	52.7	251 38.5	20.6	52 12.8	08.8	127 36.1	87	28 28.9	− 20
17	240 12.9	91 44.0	54.2	15 55.4	53.1	266 41.3	20.7	67 14.9	08.7	142 03.8	87	28 26.9	− 23
18	255 15.3	106 43.5	+13 53.2	30 56.2	−17 53.6	281 44.1	−12 20.8	82 17.1	+ 5 08.5	156 31.5	87	+28 24.6	− 24
19	270 17.8	121 42.9	52.2	45 57.1	54.1	296 46.8	20.9	97 19.3	08.4	170 59.2	87	28 22.2	− 25
20	285 20.3	136 42.4	51.3	60 57.9	54.6	311 49.6	21.1	112 21.5	08.3	185 26.9	87	28 19.7	− 27
21	300 22.7	151 41.8	·· 50.3	75 58.7	·· 55.1	326 52.3	·· 21.2	127 23.6	·· 08.2	199 54.6	87	·28 17.0	− 28
22	315 25.2	166 41.3	49.3	90 59.5	55.6	341 55.1	21.3	142 25.8	08.0	214 22.3	87	28 14.2	− 30
23	330 27.7	181 40.8	48.3	106 00.3	56.1	356 57.9	21.4	157 28.0	07.9	228 50.0	87	28 11.2	− 31
24	345 30.1	196 40.2	+13 47.3	121 01.1	−17 56.6	12 00.6	−12 21.5	172 30.2	+ 5 07.8	243 17.7	86	+28 08.1	− 33
code	—	− 6	− 10	+ 8	− 5	+ 28	− 1	+ 22	− 1				

FIGURE 3

GCT	SUN GHA	SUN Dec.
h	° ′	° ′
0	180 10.6	+ 7 29.0
1	195 10.8	28.1
2	210 11.0	27.2
3	225 11.2 ··	26.3
4	240 11.4	25.4
5	255 11.6	24.4
6	270 11.8	+ 7 23.5
7	285 12.0	22.6
8	300 12.2	21.7
9	315 12.4 ··	20.8
10	330 12.6	19.8
11	345 12.8	18.9
12	0 13.0	+ 7 18.0
13	15 13.2	17.1
14	30 13.4	16.2
15	45 13.6 ··	15.2
16	60 13.9	14.3
17	75 14.1	13.4
18	90 14.3	+ 7 12.5
19	105 14.5	11.6
20	120 14.7	10.6
21	135 14.9 ··	09.7
22	150 15.1	08.8
23	165 15.3	07.9
0	180 15.5	+ 7 06.9
1	195 15.7	06.0
2	210 15.9	05.1
3	225 16.1 ··	04.2
4	240 16.3	03.2
5	255 16.5	02.3
6	270 16.7	+ 7 01.4
7	285 16.9	7 00.5
8	300 17.1	6 59.5
9	315 17.4 ··	58.6
10	330 17.6	57.7
11	345 17.8	56.8
12	0 18.0	+ 6 55.8
13	15 18.2	54.9
14	30 18.4	54.0
15	45 18.6 ··	53.1
16	60 18.8	52.1
17	75 19.0	51.2
18	90 19.2	+ 6 50.3
19	105 19.4	49.3
20	120 19.6	48.4
21	135 19.8 ··	47.5
22	150 20.0	46.6
23	165 20.3	45.6
0	180 20.5	+ 6 44.7
1	195 20.7	43.8
2	210 20.9	42.8
3	225 21.1 ··	41.9
4	240 21.3	41.0
5	255 21.5	40.1
6	270 21.7	+ 6 39.1
7	285 21.9	38.2
8	300 22.1	37.3
9	315 22.3 ··	36.3
10	330 22.5	35.4
11	345 22.8	34.5
12	0 23.0	+ 6 33.6
13	15 23.2	32.6
14	30 23.4	31.7
15	45 23.6 ··	30.8
16	60 23.8	29.8
17	75 24.0	28.9
18	90 24.2	+ 6 28.0
19	105 24.4	27.0
20	120 24.6	26.1
21	135 24.9 ··	25.2
22	150 25.1	24.2
23	165 25.3	23.3
24	180 25.5	+ 6 22.4
code	—	− 9

STARS No.	SHA	Dec.	Transit Mer. Gr.
	° ′	° ′	h m
1	358 32.3	+28 49.2	1 12
2	358 21.7	+58 52.6	1 12
3	349 42.9	−18 15.2	1 47
4	339 21.2	+59 58.7	2 28
5	336 01.1	−57 28.9	2 41
6	332 36.5	+89 01.8	2 55
7	328 54.0	+23 13.9	3 10
8	315 53.9	−40 29.7	4 02
9	309 48.1	+49 41.2	4 26
10	291 43.7	+16 24.9	5 38
11	281 57.5	− 8 15.1	6 17
12	281 44.5	+45 57.0	6 18
13	279 22.8	+ 6 18.6	6 27
14	279 12.5	+28 34.2	6 28
15	276 34.4	− 1 13.6	6 38
16	271 52.6	+ 7 24.2	6 57
17	264 17.4	−52 39.6	7 27
18	259 15.5	−16 38.5	7 48
19	255 49.9	−28 53.8	8 01
20	245 49.4	+ 5 21.4	8 41
21	244 25.7	+28 09.0	8 47
22	234 38.1	−59 20.7	9 26
23	223 27.8	−43 13.7	10 10
24	221 51.1	−69 30.6	10 17
25	218 42.9	− 8 26.4	10 29
26	208 34.1	+12 12.7	11 10
27	194 49.9	+62 01.2	12 05
28	183 22.1	+14 51.0	12 50
29	174 03.3	−62 49.7	13 27
30	172 54.5	−56 50.3	13 32
31	168 48.4	−59 25.3	13 48
32	167 02.3	+56 13.8	13 55
33	159 31.1	+55 11.1	14 25
34	159 21.4	−10 54.2	14 26
35	153 36.2	+49 33.7	14 49
36	149 03.8	−36 07.8	15 07
37	146 39.0	+19 26.4	15 17
38	140 57.1	−60 38.3	15 40
39	137 17.8	+74 21.6	15 54
40	126 51.1	+26 52.9	16 36
41	120 38.9	−22 29.1	17 00
42	113 24.4	−26 19.6	17 29
43	109 08.8	−68 56.8	17 46
44	103 06.8	−15 40.0	18 10
45	97 26.1	−37 04.4	18 33
46	96 50.3	+12 35.7	18 35
47	91 07.9	+51 29.8	18 58
48	84 46.3	−34 24.8	19 24
49	81 10.8	+38 44.3	19 38
50	76 56.7	−26 21.7	19 55
51	62 54.1	+ 8 44.2	20 51
52	54 33.0	−56 53.9	21 24
53	50 03.5	+45 06.3	21 42
54	34 33.3	+ 9 38.9	22 44
55	28 42.4	−47 12.1	23 07
56	16 15.6	−29 53.0	0 01
57	14 25.2	+14 56.5	0 08

GCT	Equation of Time (App.—Mean) 4	5	6
h	m s	m s	m s
0	+ 0 41.9	+ 1 01.6	+ 1 21.4
6	+ 0 46.8	+ 1 06.5	+ 1 26.4
12	+ 0 51.7	+ 1 11.5	+ 1 31.4
18	+ 0 56.6	+ 1 16.4	+ 1 36.5
24	+ 1 01.6	+ 1 21.4	+ 1 41.5

Lat.	Sunrise	Twlt. begins	Moonrise 4	5	6	7
N °	h m	h m	h m	h m	h m	h m
70	4 32	////	□	□	□	□
68	4 41	////	□	□	□	□
66	4 48	////	□	□	□	□
64	4 54	1 13	□	□	□	□
62	4 59	1 51	18 44	□	□	22 02
60	5 04	2 16	19 38	20 05	21 08	22 42
58	5 08	2 35	20 11	20 46	21 47	23 10
56	5 12	2 51	20 35	21 14	22 13	23 31
54	5 15	3 04	20 55	21 36	22 35	23 49
52	5 18	3 15	21 11	21 54	22 52	24 04
50	5 21	3 24	21 25	22 09	23 07	24 17
45	5 27	3 43	21 54	22 41	23 37	24 44
40	5 31	3 57	22 17	23 05	24 01	0 01
35	5 35	4 08	22 36	23 25	24 20	0 20
30	5 39	4 17	22 52	23 41	24 37	0 37
20	5 45	4 31	23 20	24 10	0 10	1 05
10	5 51	4 41	23 43	24 35	0 35	1 29
0	5 55	4 46	24 06	0 06	0 58	1 51
10	6 00	4 50	24 28	0 28	1 21	2 14
20	6 05	4 52	24 52	0 52	1 46	2 38
30	6 11	4 52	0 22	1 20	2 14	3 06
35	6 14	4 50	0 37	1 36	2 32	3 22
40	6 17	4 47	0 55	1 55	2 51	3 41
45	6 21	4 43	1 16	2 19	3 16	4 04
50	6 26	4 38	1 43	2 49	3 47	4 34
52	6 28	4 35	1 56	3 04	4 03	4 49
54	6 31	4 33	2 11	3 22	4 21	5 05
56	6 33	4 29	2 28	3 43	4 43	5 26
58	6 36	4 25	2 50	4 10	5 12	5 51
60	6 39	4 20	3 17	4 49	5 56	6 26
S						

Lat.	Sunset	Twlt. ends	Moonset 4	5	6	7
N °	h m	h m	h m	h m	h m	h m
70	19 23	////	□	□	□	□
68	19 14	////	□	□	□	□
66	19 07	////	□	□	□	□
64	19 01	22 42	□	□	□	□
62	18 56	22 04	16 55	□	□	19 08
60	18 52	21 40	16 01	17 22	18 10	18 27
58	18 48	21 21	15 28	16 41	17 31	17 59
56	18 44	21 05	15 04	16 13	17 04	17 38
54	18 41	20 52	14 45	15 51	16 43	17 20
52	18 38	20 41	14 29	15 33	16 25	17 04
50	18 35	20 32	14 15	15 18	16 10	16 51
45	18 30	20 14	13 47	14 47	15 39	16 23
40	18 25	19 59	13 24	14 23	15 16	16 02
35	18 21	19 48	13 06	14 03	14 56	15 44
30	18 18	19 40	12 50	13 46	14 39	15 28
20	18 12	19 26	12 23	13 18	14 11	15 02
10	18 07	19 17	12 00	12 53	13 46	14 38
0	18 02	19 11	11 39	12 30	13 24	14 17
10	17 57	19 07	11 18	12 08	13 01	13 55
20	17 53	19 06	10 55	11 43	12 36	13 32
30	17 47	19 06	10 29	11 15	12 07	13 05
35	17 44	19 08	10 13	10 58	11 50	12 49
40	17 41	19 11	9 55	10 38	11 30	12 31
45	17 37	19 15	9 34	10 15	11 06	12 08
50	17 32	19 20	9 06	9 44	10 35	11 39
52	17 30	19 23	8 53	9 29	10 19	11 24
54	17 28	19 26	8 37	9 11	10 01	11 08
56	17 25	19 29	8 19	8 50	9 38	10 48
58	17 22	19 33	7 58	8 22	9 09	10 23
60	17 19	19 38	7 29	7 43	8 26	9 48
S						

GCT of Transit Meridian of Greenwich

DAY	Moon	Venus	Mars	Jupiter	Saturn
	h m	h m	h m	h m	h m
4	5 27	10 51	15 59	23 21	12 39
5	6 18	10 52	15 58	23 17	12 35
6	7 11	10 53	15 56	23 12	12 32
7	8 04	10 54	15 55	23 08	12 28

ADDITIONAL ALTITUDE CORRECTIONS

Alt.	Sun's Low'r Limb	Venus	Mars
°	′	′	′
0			
	0.2	0.1	0.1
30			
	0.2	0.1	0.1
45			
	0.2	0.1	0.1
60			
	0.2	0.0	0.0
75			
	0.2	0.0	0.0
90			

Moon's Lower Limb *Add*

Alt.	4	5	6
°	′	′	′
0			
	12.1	12.5	13.0
4			
	12.1	12.5	13.0
8			
	12.2	12.5	13.0
12			
	12.2	12.6	13.1
16			
	12.2	12.6	13.1
20			
	12.3	12.7	13.2
24			
	12.4	12.7	13.2
28			
	12.5	12.8	13.3
32			
	12.6	12.9	13.3
36			
	12.7	13.0	13.4
40			
	12.8	13.1	13.5
44			
	12.9	13.2	13.6
48			
	13.1	13.4	13.7
52			
	13.2	13.5	13.8
56			
	13.4	13.6	13.9
60			
	13.5	13.8	14.1
64			
	13.7	13.9	14.2
68			
	13.9	14.1	14.3
72			
	14.1	14.2	14.5
76			
	14.3	14.4	14.6
80			
	14.4	14.6	14.7
84			
	14.6	14.7	14.9
88			
	14.8	14.9	15.0
90			

Add to table, back cover

Moon's Upper Limb *Subtract*

Alt.	4	5	6
°	′	′	′
0			
	17.5	17.3	17.0
4			
	17.5	17.3	17.0
8			
	17.5	17.3	17.0
12			
	17.4	17.2	16.9
16			
	17.4	17.2	16.9
20			
	17.3	17.1	16.9
24			
	17.2	17.1	16.8
28			
	17.2	17.0	16.7
32			
	17.1	16.9	16.7
36			
	16.9	16.8	16.6
40			
	16.8	16.7	16.5
44			
	16.7	16.6	16.4
48			
	16.6	16.4	16.3
52			
	16.4	16.3	16.2
56			
	16.2	16.2	16.1
60			
	16.1	16.0	15.9
64			
	15.9	15.9	15.8
68			
	15.7	15.7	15.7
72			
	15.6	15.6	15.6
76			
	15.4	15.4	15.4
80			
	15.2	15.2	15.3
84			
	15.0	15.1	15.1
88			
	14.9	14.9	15.0
90			

Subtract from table, back cover

Labor Day 4
Last Quarter 4

FIGURE 4

264 12m

Sec.	SUN PLANET	♈	MOON	code	corr'n	code	corr'n	code	corr'n
	° ′	° ′	° ′		′		′		′
0	3 00.0	3 00.5	2 51.8	0	0.0	60	1.3	120	2.5
1	3 00.3	3 00.7	2 52.0	1	0.0	61	1.3	121	2.5
2	3 00.5	3 01.0	2 52.3	2	0.0	62	1.3	122	2.5
3	3 00.8	3 01.2	2 52.5	3	0.1	63	1.3	123	2.6
4	3 01.0	3 01.5	2 52.8	4	0.1	64	1.3	124	2.6
5	3 01.3	3 01.7	2 53.0	5	0.1	65	1.4	125	2.6
6	3 01.5	3 02.0	2 53.2	6	0.1	66	1.4	126	2.6
7	3 01.8	3 02.2	2 53.5	7	0.1	67	1.4	127	2.6
8	3 02.0	3 02.5	2 53.7	8	0.2	68	1.4	128	2.7
9	3 02.3	3 02.7	2 53.9	9	0.2	69	1.4	129	2.7
10	3 02.5	3 03.0	2 54.2	10	0.2	70	1.5	130	2.7
11	3 02.8	3 03.3	2 54.4	11	0.2	71	1.5	131	2.7
12	3 03.0	3 03.5	2 54.7	12	0.3	72	1.5	132	2.8
13	3 03.3	3 03.8	2 54.9	13	0.3	73	1.5	133	2.8
14	3 03.5	3 04.0	2 55.1	14	0.3	74	1.5	134	2.8
15	3 03.8	3 04.3	2 55.4	15	0.3	75	1.6	135	2.8
16	3 04.0	3 04.5	2 55.6	16	0.3	76	1.6	136	2.8
17	3 04.3	3 04.8	2 55.9	17	0.4	77	1.6	137	2.9
18	3 04.5	3 05.0	2 56.1	18	0.4	78	1.6	138	2.9
19	3 04.8	3 05.3	2 56.3	19	0.4	79	1.6	139	2.9
20	3 05.0	3 05.5	2 56.6	20	0.4	80	1.7	140	2.9
21	3 05.3	3 05.8	2 56.8	21	0.4	81	1.7	141	2.9
22	3 05.5	3 06.0	2 57.0	22	0.5	82	1.7	142	3.0
23	3 05.8	3 06.3	2 57.3	23	0.5	83	1.7	143	3.0
24	3 06.0	3 06.5	2 57.5	24	0.5	84	1.8	144	3.0
25	3 06.3	3 06.8	2 57.8	25	0.5	85	1.8	145	3.0
26	3 06.5	3 07.0	2 58.0	26	0.5	86	1.8	146	3.0
27	3 06.8	3 07.3	2 58.2	27	0.6	87	1.8	147	3.1
28	3 07.0	3 07.5	2 58.5	28	0.6	88	1.8	148	3.1
29	3 07.3	3 07.8	2 58.7	29	0.6	89	1.9	149	3.1
30	3 07.5	3 08.0	2 59.0	30	0.6	90	1.9	150	3.1
31	3 07.8	3 08.3	2 59.2	31	0.6	91	1.9	151	3.1
32	3 08.0	3 08.5	2 59.4	32	0.7	92	1.9	152	3.2
33	3 08.3	3 08.8	2 59.7	33	0.7	93	1.9	153	3.2
34	3 08.5	3 09.0	2 59.9	34	0.7	94	2.0	154	3.2
35	3 08.8	3 09.3	3 00.2	35	0.7	95	2.0	155	3.2
36	3 09.0	3 09.5	3 00.4	36	0.8	96	2.0	156	3.3
37	3 09.3	3 09.8	3 00.6	37	0.8	97	2.0	157	3.3
38	3 09.5	3 10.0	3 00.9	38	0.8	98	2.0	158	3.3
39	3 09.8	3 10.3	3 01.1	39	0.8	99	2.1	159	3.3
40	3 10.0	3 10.5	3 01.3	40	0.8	100	2.1	160	3.3
41	3 10.3	3 10.8	3 01.6	41	0.9	101	2.1	161	3.4
42	3 10.5	3 11.0	3 01.8	42	0.9	102	2.1	162	3.4
43	3 10.8	3 11.3	3 02.1	43	0.9	103	2.1	163	3.4
44	3 11.0	3 11.5	3 02.3	44	0.9	104	2.2	164	3.4
45	3 11.3	3 11.8	3 02.5	45	0.9	105	2.2	165	3.4
46	3 11.5	3 12.0	3 02.8	46	1.0	106	2.2	166	3.5
47	3 11.8	3 12.3	3 03.0	47	1.0	107	2.2	167	3.5
48	3 12.0	3 12.5	3 03.3	48	1.0	108	2.3	168	3.5
49	3 12.3	3 12.8	3 03.5	49	1.0	109	2.3	169	3.5
50	3 12.5	3 13.0	3 03.7	50	1.0	110	2.3	170	3.5
51	3 12.8	3 13.3	3 04.0	51	1.1	111	2.3	171	3.6
52	3 13.0	3 13.5	3 04.2	52	1.1	112	2.3	172	3.6
53	3 13.3	3 13.8	3 04.4	53	1.1	113	2.4	173	3.6
54	3 13.5	3 14.0	3 04.7	54	1.1	114	2.4	174	3.6
55	3 13.8	3 14.3	3 04.9	55	1.1	115	2.4	175	3.6
56	3 14.0	3 14.5	3 05.2	56	1.2	116	2.4	176	3.7
57	3 14.3	3 14.8	3 05.4	57	1.2	117	2.4	177	3.7
58	3 14.5	3 15.0	3 05.6	58	1.2	118	2.5	178	3.7
59	3 14.8	3 15.3	3 05.9	59	1.2	119	2.5	179	3.7
60	3 15.0	3 15.5	3 06.1	60	1.3	120	2.5	180	3.8

13m

Sec.	SUN PLANET	♈	MOON	code	corr'n	code	corr'n	code	corr'n
	° ′	° ′	° ′		′		′		′
0	3 15.0	3 15.5	3 06.1	0	0.0	60	1.4	120	2.7
1	3 15.3	3 15.8	3 06.4	1	0.0	61	1.4	121	2.7
2	3 15.5	3 16.0	3 06.6	2	0.0	62	1.4	122	2.7
3	3 15.8	3 16.3	3 06.8	3	0.1	63	1.4	123	2.8
4	3 16.0	3 16.5	3 07.1	4	0.1	64	1.4	124	2.8
5	3 16.3	3 16.8	3 07.3	5	0.1	65	1.5	125	2.8
6	3 16.5	3 17.0	3 07.5	6	0.1	66	1.5	126	2.8
7	3 16.8	3 17.3	3 07.8	7	0.2	67	1.5	127	2.9
8	3 17.0	3 17.5	3 08.0	8	0.2	68	1.5	128	2.9
9	3 17.3	3 17.8	3 08.3	9	0.2	69	1.6	129	2.9
10	3 17.5	3 18.0	3 08.5	10	0.2	70	1.6	130	2.9
11	3 17.8	3 18.3	3 08.7	11	0.2	71	1.6	131	2.9
12	3 18.0	3 18.5	3 09.0	12	0.3	72	1.6	132	3.0
13	3 18.3	3 18.8	3 09.2	13	0.3	73	1.6	133	3.0
14	3 18.5	3 19.0	3 09.5	14	0.3	74	1.7	134	3.0
15	3 18.8	3 19.3	3 09.7	15	0.3	75	1.7	135	3.0
16	3 19.0	3 19.5	3 09.9	16	0.4	76	1.7	136	3.1
17	3 19.3	3 19.8	3 10.2	17	0.4	77	1.7	137	3.1
18	3 19.5	3 20.0	3 10.4	18	0.4	78	1.8	138	3.1
19	3 19.8	3 20.3	3 10.7	19	0.4	79	1.8	139	3.1
20	3 20.0	3 20.5	3 10.9	20	0.5	80	1.8	140	3.2
21	3 20.3	3 20.8	3 11.1	21	0.5	81	1.8	141	3.2
22	3 20.5	3 21.0	3 11.4	22	0.5	82	1.8	142	3.2
23	3 20.8	3 21.3	3 11.6	23	0.5	83	1.9	143	3.2
24	3 21.0	3 21.6	3 11.8	24	0.5	84	1.9	144	3.2
25	3 21.3	3 21.8	3 12.1	25	0.6	85	1.9	145	3.3
26	3 21.5	3 22.1	3 12.3	26	0.6	86	1.9	146	3.3
27	3 21.8	3 22.3	3 12.6	27	0.6	87	2.0	147	3.3
28	3 22.0	3 22.6	3 12.8	28	0.6	88	2.0	148	3.3
29	3 22.3	3 22.8	3 13.0	29	0.7	89	2.0	149	3.4
30	3 22.5	3 23.1	3 13.3	30	0.7	90	2.0	150	3.4
31	3 22.8	3 23.3	3 13.5	31	0.7	91	2.0	151	3.4
32	3 23.0	3 23.6	3 13.8	32	0.7	92	2.1	152	3.4
33	3 23.3	3 23.8	3 14.0	33	0.7	93	2.1	153	3.4
34	3 23.5	3 24.1	3 14.2	34	0.8	94	2.1	154	3.5
35	3 23.8	3 24.3	3 14.5	35	0.8	95	2.1	155	3.5
36	3 24.0	3 24.6	3 14.7	36	0.8	96	2.2	156	3.5
37	3 24.3	3 24.8	3 14.9	37	0.8	97	2.2	157	3.5
38	3 24.5	3 25.1	3 15.2	38	0.9	98	2.2	158	3.6
39	3 24.8	3 25.3	3 15.4	39	0.9	99	2.2	159	3.6
40	3 25.0	3 25.6	3 15.7	40	0.9	100	2.3	160	3.6
41	3 25.3	3 25.8	3 15.9	41	0.9	101	2.3	161	3.6
42	3 25.5	3 26.1	3 16.1	42	0.9	102	2.3	162	3.6
43	3 25.8	3 26.3	3 16.4	43	1.0	103	2.3	163	3.7
44	3 26.0	3 26.6	3 16.6	44	1.0	104	2.3	164	3.7
45	3 26.3	3 26.8	3 16.9	45	1.0	105	2.4	165	3.7
46	3 26.5	3 27.1	3 17.1	46	1.0	106	2.4	166	3.7
47	3 26.8	3 27.3	3 17.3	47	1.1	107	2.4	167	3.8
48	3 27.0	3 27.6	3 17.6	48	1.1	108	2.4	168	3.8
49	3 27.3	3 27.8	3 17.8	49	1.1	109	2.5	169	3.8
50	3 27.5	3 28.1	3 18.0	50	1.1	110	2.5	170	3.8
51	3 27.8	3 28.3	3 18.3	51	1.1	111	2.5	171	3.8
52	3 28.0	3 28.6	3 18.5	52	1.2	112	2.5	172	3.9
53	3 28.3	3 28.8	3 18.8	53	1.2	113	2.5	173	3.9
54	3 28.5	3 29.1	3 19.0	54	1.2	114	2.6	174	3.9
55	3 28.8	3 29.3	3 19.2	55	1.2	115	2.6	175	3.9
56	3 29.0	3 29.6	3 19.5	56	1.3	116	2.6	176	4.0
57	3 29.3	3 29.8	3 19.7	57	1.3	117	2.6	177	4.0
58	3 29.5	3 30.1	3 20.0	58	1.3	118	2.7	178	4.0
59	3 29.8	3 30.3	3 20.2	59	1.3	119	2.7	179	4.0
60	3 30.0	3 30.6	3 20.4	60	1.4	120	2.7	180	4.1

FIGURE 5

APPARENT DECLINATIONS OF STAR-PAIRS

For the upper transit, meridian of La Plata

Greenwich mean astronomical dates

Group X

		55		56		57		58		59		60	
		° ′		° ′		° ′		° ′		° ′		° ′	
1947		−34 51		−34 52		−34 57		−34 54		−34 51		−34 51	
		″		″		″		″		″		″	
6	5	41.85	5	54.44	5	16.82	3	41.37	2	26.21	0	15.35	1
	6	41.90	8	54.49	6	16.85	6	41.39	4	26.21	3	15.34	0
	7	41.98	9	54.55	8	16.91	5	41.43	4	26.24	3	15.34	1
	8	42.07	9	54.63	8	16.96	8	41.47	6	26.27	5	15.35	3
	9	42 16	9	54.71	8	17.04	7	41.53	6	26.32	4	15.38	3
	10	42.25	9	54.79	8	17.11	7	41.59	6	26.36	5	15.41	3
	11	42.34	7	54.87	6	17.18	5	41.65	5	26.41	4	15.44	3
	12	42.41	4	54.93	3	17.23	3	41.70	2	26.45	1	15.47	1
	13	42.45	2	54.96	2	17.26	1	41.72	0	26.46	1	15.46	1
	14	42.47	2	54.98	2	17.27	0	41.72	1	26.45	2	15.45	4
	15	42.49	2	55.00	0	17.27	1	41.71	2	26.43	4	15.41	5
	16	42.51	1	55.00	0	17.26	0	41.69	2	26.39	2	15.36	4
	17	42.52	4	55.00	3	17.26	2	41.67	1	26.37	2	15.32	4
	18	42.56	8	55.03	6	17.28	4	41.68	3	26.35	2	15.28	0
	19	42.64	8	55.09	7	17.32	7	41.71	5	26.37	4	15.28	2
	20	42.72	11	55.16	11	17.39	9	41.76	8	26.41	6	15.30	4
	21	42.83	12	55.27	11	17.48	10	41.84	9	26.47	7	15.34	6
	22	42.95	12	55.38	11	17.58	10	41.93	9	26.54	8	15.40	6
	23	43.07	9	55.49	8	17.68	8	42.02	7	26.62	7	15.46	6
	24	43.16	7	55.57	7	17.76	6	42.09	6	26.69	5	15.52	4
	25	43.23	6	55.64	5	17.82	5	42.15	4	26.74	3	15.56	3
	26	43.29	4	55.69	4	17.87	3	42.19	3	26.77	2	15.59	1
	27	43.33	3	55.73	2	17.90	2	42.22	1	26.79	1	15.60	0
	28	43.36	3	55.75	3	17.92	2	42.23	1	26.80	0	15.60	1
	29	43.39	4	55.78	3	17.94	3	42.24	2	26.80	0	15.59	1
	30	43.43	6	55.81	5	17.97	4	42.26	4	26.80	3	15.58	1
7	1	43.49	7	55.86	6	18.01	6	42.30	4	26.83	3	15.59	3
	2	43.56	8	55.92	8	18.07	6	42.34	6	26.86	4	15.62	3
	3	43.64	10	56.00	9	18.13	9	42.40	8	26.90	7	15.65	5
	4	43.74	11	56.09	10	18.22	9	42.48	9	26.97	8	15.70	7
	5	43.85	11	56.19	11	18.31	11	42.57	9	27.05	8	15.77	8
	6	43.96	11	56.30	11	18.42	10	42.66	10	27.13	9	15.85	8
	7	44.07		56.41		18.52		42.76		27.22		15.93	

FIGURE 6

917 Lyka — $14^{m}5$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 4		12 36.0	6.3	− 6 23	24	214°
12		12 29.7	7.2	− 5 59	31	0.448
20	26	12 22.5	7.8	− 5 28	36	−66′
28		12 14.7	7.7	− 4 52	38	$8^{m}3$
4 5		12 07.0	7.0	− 4 14	37	0.260
13		12 00.0		− 3 37		1*

278 Paulina — $12^{m}0$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 4		12 39.1	5.3	+ 8 32	39	346°
12		12 33.8	6.4	+ 9 11	36	0.381
20	27	12 27.4	7.0	+ 9 47	30	−7.′6
28		12 20.4	7.0	+10 17	20	
4 5		12 13.4	6.2	+10 37	7	0.153
13		12 07.2		+10 44		2*

217 Eudora — $13^{m}8$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 4		12 39.7	4.8	+ 0 32	55	260°
12		12 34.9	5.6	+ 1 27	59	0.514
20	27	12 29.3	6.1	+ 2 26	60	−3.′6
28		12 23.2	6.0	+ 3 26	58	
4 5		12 17.2	5.9	+ 4 24	53	0.357
13		12 11.3		+ 5 17		2*

1486 Marilyn — $15^{m}6$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 12		12 39.7	7.2	− 4 19	44	235°
20		12 32.5	7.9	− 3 35	50	0.376
28	28	12 24.6	8.0	− 2 45	52	−6.′5
4 5		12 16.6	7.5	− 1 53	49	
13		12 09.1	6.4	− 1 04	40	0.139
21		12 02.7		− 0 24		13

1485 Isa — $15^{m}5$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 4		12 41.4	5.4	−16 38	7	220°
12		12 36.0	5.9	−16 31	17	0.520
20	28	12 30.1	6.2	−16 14	26	−47′
28		12 23.9	6.4	−15 48	34	$6^{m}6$
4 5		12 17.5	5.9	−15 14	38	0.368
13		12 11.6		−14 36		7*

1496 1938 SA_1 — $16^{m}2$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 4		12 44.3	5.7	− 8 53	23	272°
12		12 38.6	7.1	− 8 30	34	0.352
20	28	12 31.5	8.0	− 7 56	43	−6.′7
28		12 23.5	8.2	− 7 13	49	
4 5		12 15.3	7.7	− 6 24	50	0.098
13		12 07.6		− 5 34		12*

511 Davida — $9^{m}7$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 12		12 39.6	5.6	+18 45	55	85°
20		12 34.0	5.9	+19 40	44	0.510
28	29	12 28.1	5.9	+20 24	32	−37′
4 5		12 22.2	5.5	+20 56	18	$7^{m}6$
13		12 16.7	4.7	+21 14	5	0.360
21		12 12.0		+21 19		1*

385 Ilmatar — $9^{m}6$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 12		12 43.4	7.1	−12 50	16	13°
20		12 36.3	8.0	−13 06	6	0.399
28	29	12 28.3	8.0	−13 12	3	−9.′8
4 5		12 20.3	7.5	−13 09	8	
13		12 12.8	6.5	−13 01	11	0.180
21		12 06.3		−12 50		2*

929 Algunde — $13^{m}3$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 12		12 41.8	6.2	−10 15	44	307°
20		12 35.6	7.2	− 9 31	55	0.324
28	29	12 28.4	7.4	− 8 36	63	−107′
4 5		12 21.0	6.9	− 7 33	65	$18^{m}6$
13		12 14.1	5.7	− 6 28	62	0.046
21		12 08.4		− 5 26		1*

1291 Phryne — $13^{m}8$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 12		12 40.2	5.1	−11 15	42	229°
20		12 35.1	5.7	−10 33	49	0.507
28	29	12 29.4	5.8	− 9 44	54	−4.′3
4 5		12 23.6	5.5	− 8 50	56	
13		12 18.1	4.8	− 7 54	54	0.347
21		12 13.3		− 7 00		13

1530 1938 SG — $17^{m}7$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 12		12 44.6	7.1	−12 02	33	180°
20		12 37.5	7.8	−11 29	42	0.431
28	29	12 29.7	7.9	−10 47	49	−6.′5
4 5		12 21.8	7.4	− 9 58	52	
13		12 14.4	6.6	− 9 06	50	0.232
21		12 07.8		− 8 16		12*

1549 Mikko — $15^{m}2$

1947		α_{1950} (h m)		δ_{1950} (° ′)		Misc.
3 12		12 45.3	7.2	+ 5 47	53	85°
20		12 38.1	7.9	+ 6 40	49	0.349
28	29	12 30.2	7.9	+ 7 29	39	−6.′8
4 5		12 22.3	7.1	+ 8 08	26	
13		12 15.2	5.9	+ 8 34	12	0.095
21		12 09.3		+ 8 46		12*

FIGURE 7

	No.	m	g	Epoch 0^hUT			M	ω	Ω 1950.0	i	φ	n	a
		m	m				°	°	°	°	°	"	
	601	12.6	8.5	1943	III	17	178.170	158.150	170.066	16.101	6.262	639.412	3.1344
	602	12.1	8.0	1938	I	20	71.960	41.999	332.805	15.215	13.647	649.958	3.1003
	603	13.9	10.9	1900	I	0	182.506	154.127	344.283	8.028	9.822	871.097	2.5505
	604	12.4	8.2	1945	I	25	67.547	26.630	12.980	4.433	10.421	630.418	3.1641
	605	12.9	9.0	1937	X	24	38.853	12.440	343.278	19.666	7.769	682.398	3.0013
	606	12.9	9.8	1900	I	0	134.016	54.312	319.949	8.683	12.681	852.425	2.5877
	607	12.6	9.0	1918	II	19	280.374	288.700	286.257	10.088	4.475	736.802	2.8517
	608	14.1	10.2	1942	VII	20	297.605	67.437	294.668	9.374	6.790	674.113	3.0259
	609	12.8	8.8	1934	V	31	112.365	122.275	166.265	4.147	2.026	653.676	3.0886
	610	15.6	11.6	1942	V	1	209.601	350.997	20.990	13.088	14.763	656.500	3.0797
	611	12.3	8.4	1939	II	22	51.990	254.359	190.333	13.407	6.973	689.747	2.9800
	612	14.6	10.4	1906	X	9	24.381	116.310	205.785	20.492	15.462	636.959	3.1424
	613	13.0	9.3	1945	VIII	13	254.689	63.224	355.185	7.670	3.583	711.389	2.9192
	614	13.7	10.4	1919	IX	3	299.931	206.582	218.019	6.996	6.299	802.264	2.6944
A	615	12.6	9.4	1900	I	0	270.150	242.400	14.650	2.770	6.390	831.146	2.6316
	616	12.7	9.7	1900	I	0	49.570	105.860	356.660	15.000	3.410	869.943	2.5527
S	617	12.6	5.9	1940	X	9	353.645	303.410	43.934	22.103	8.130	299.717	5.1943
	618	12.4	8.2	1945	VII	20	307.744	244.184	111.473	17.007	4.739	623.700	3.1868
	619	12.1	9.2	1900	I	0	142.400	174.600	188.420	13.740	4.370	886.799	2.5203
	620	13.6	10.9	1900	I	0	129.740	333.090	0.860	7.770	7.660	933.328	2.4358
	621	13.9	9.8	1942	V	1	129.029	30.373	67.485	2.357	7.883	641.457	3.1277
	622	12.8	10.1	1917	IX	15	329.798	254.035	142.956	8.641	14.032	945.316	2.4152
	623	12.8	10.0	1900	I	0	111.540	123.030	309.120	14.170	6.550	919.333	2.4604
S	624	13.2	6.4	1940	XII	19	293.458	177.001	342.152	18.267	1.613	304.721	5.1373
	625	12.1	8.9	1950	I	0	168.883	198.759	127.748	12.093	12.977	823.989	2.6469
	626	11.4	8.4	1922	XII	15	35.555	42.387	342.021	25.454	14.041	859.549	2.5733
	627	13.1	9.3	1933	V	21	293.478	177.613	143.053	6.449	3.383	718.676	2.8995
A	628	12.2	9.2	1900	I	0	294.050	201.720	112.690	11.541	2.453	855.232	2.5820
	629	13.8	9.7	1946	I	20	23.677	31.984	87.746	9.322	8.819	641.364	3.1280
	630	13.5	10.3	1950	I	0	35.080	37.780	105.710	13.900	6.500	834.894	2.6237
	631	12.3	8.8	1921	IV	28	68.920	276.906	225.621	18.830	4.811	760.172	2.7929
	632	14.5	11.3	1900	I	0	97.310	247.060	358.770	2.262	11.080	816.653	2.6626
	633	12.9	9.0	1937	II	1	153.372	188.894	147.937	10.906	4.947	676.596	3.0184
	634	13.1	9.1	1937	II	1	113.825	220.407	134.084	12.288	10.494	666.462	3.0489
S	635	12.6	8.5	1944	I	19	58.589	227.295	183.961	10.979	4.688	637.307	3.1413
	636	12.4	8.7	1937	III	10	185.976	296.589	35.429	7.939	9.975	714.847	2.9098
	637	14.0	9.8	1941	III	21	12.612	164.666	357.000	0.324	6.799	631.934	3.1591
	638	13.5	10.1	1943	III	28	337.124	127.194	103.655	7.708	9.182	784.808	2.7342
	639	12.1	8.2	1943	V	25	271.534	65.216	280.666	8.559	6.327	678.516	3.0127
	640	13.0	8.8	1936	XI	13	149.283	18.737	236.156	13.374	3.923	630.670	3.1632
	641	14.5	12.3	1925	I	1	30.102	16.416	41.106	1.733	7.392	1072.666	2.2200
	642	13.5	9.3	1934	I	13	122.898	110.525	7.776	8.193	8.481	627.618	3.1735
S	643	13.9	9.4	1945	XII	16	352.206	210.628	255.110	13.902	5.485	581.259	3.3401
	644	13.1	10.0	1900	I	0	66.090	267.510	108.940	1.040	8.980	846.504	2.5997
	645	13.5	9.3	1939	V	5	119.839	87.782	0.990	7.063	9.920	623.667	3.1869
	646	14.5	12.1	1950	I	0	346.145	36.155	303.047	6.945	12.327	1000.813	2.3251
	647	13.5	10.8	1939	I	15	24.619	173.639	254.826	7.299	10.956	928.740	2.4439
	648	13.1	8.9	1938	IV	7	72.117	168.389	292.933	9.989	13.102	628.555	3.1704
	649	15.1	12.1	1900	I	0	44.768	347.088	357.887	12.680	16.053	871.566	2.5495
	650	14.7	11.9	1907	X	5	3.071	175.990	216.357	2.555	10.770	918.478	2.4620

FIGURE 8

through the typewriter would permit the superposition of several curves, if desired. This problem has been solved in a general way, but some details remain to be worked out. It will be possible to process data in a Type 602, so that the resulting punches will operate in the typewriter to produce the desired plots. Possible uses of such a curve plotter, with accuracy of this order, may occur to some of you.

This has been a very brief summary of the machines and methods in use at the Nautical Almanac Office. As I have indicated, the scope of our work is continually increasing. There is considerable satisfaction to be derived from the fact that every astronomical problem to which the machines have been applied so far has been successfully solved. I have no doubt that this will continue to be the case.

REFERENCE

1. W. J. ECKERT and R. F. HAUPT, "The Printing of Mathematical Tables," *MTAC*, II (1947), pp. 197-202.

DISCUSSION

Dr. Eckert: I would like to point out that copy for the *Air Almanac* has been prepared on the typewriter for three and a half years. In that time about ten million figures were computed, typed, proofread as Mr. Hollander described, and printed in editions running as high as two hundred thousand copies. Thousands of aviators used these volumes day after day. And to date there has not been one single error reported.

Mr. Hollander: Regarding the accuracy of the typewriter, it has never been known to replace one figure with another. The worst it has ever done is fail to read the hole in the card and thus leave a blank space; this can be detected by inspection.

Dr. Eckert: This proofreading business may sound laborious, but you must remember you are putting out a publication that is going to be studied, used, and sweated over; whether the production of a figure takes ten seconds or eleven doesn't make too much difference. One girl, completely inexperienced in technical matters, holding the lowest grade clerical rating in the Civil Service, can learn in two weeks to proofread fifteen pages a day by punching it up. When one error is so important, you certainly can afford these checks.

Programming and Using the Type 603-405 Combination Machine in the Solution of Differential Equations

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FOR THE PAST two years Northrop Aircraft has used a small installation of International Business Machines equipment for engineering calculations. While much of the work was and is routine—stress distribution, reduction of wind tunnel results, and the multiplication and inversion of matrices—considerable miscellaneous work in connection with research projects has appeared from time to time. Last winter two despairing men brought us a system of differential equations for step-by-step integration.

The differential analyzer at the University of California was unable to solve the problem because of the presence of a term $a\ddot{\phi} + \phi$ where a is of the order of 10^7. The turn ratio for the $\ddot{\phi}$ and ϕ shafts made the time of solution run into centuries. Large scale digital computers were considered but the earliest schedule times were remote and the probability of needing additional work after the first set of solutions were completed introduced the possibility of another schedule delay—two years if we were lucky. After telling us this story they presented us with the system

$$\ddot{x} = a\dot{x} + by - cf_1(t)$$
$$\ddot{y} = d\dot{y} + ef_2(z)$$
$$\dot{z} = fz + g\ddot{x} + hx$$

$$f_2(z) = \begin{cases} -1 & z \geqq +\delta \\ 0 & -\delta < z < +\delta \\ +1 & z \leqq -\delta \end{cases}$$

which is not impressive except for the discontinuity introduced by the $f_2(z)$ term, which prevents analytic integration other than piece-wise. Since there are no products in which both factors vary, the problem could be integrated in the accounting machine, using repeated counter transfers for the coefficient multiplications.

Once we had this problem under way, we were given the full two dimensional problem and *were* impressed. It had almost everything—circular functions of a dependent variable as coefficients, products of variables, a couple of arbitrary input functions, and two of everything like the $f_2(z)$ in the case just described. We needed something that would read factors from the accounting machine on one cycle, multiply, and enter the product in the accounting machine on the succeeding cycle. For example, a Type 603 Electronic Multiplier hooked to the accounting machine could do the job. In addition, we needed a great many selectors for controlling counter connections; a means for converting counters for true figure read-in to the multiplier; and a balance test impulse available for every cycle for algebraic sign discrimination.

We asked the local IBM representatives for such a machine. Mr. A. B. Kimball of IBM was called in. He suggested, on first sight, going after the answer "hammer and tongs" style. For some possible time saving ideas he referred us to an account of step-by-step integration done at the Thomas J. Watson Astronomical Computing Bureau in New York by Dr. Eckert. As the problem would require more than 200,000 steps with at least six function products punched, carried to the multiplier, to the collator, and then back to the accounting machine for each cycle, this was prohibitive. The problem was then taken to IBM Headquarters in New York where Mr. J. C. McPherson went over the problem requirements in detail and agreed with our conclusion with respect to machine requirements. The machine was built. In fact, we received much more than we expected. The request was for a bare minimum of the items noted above. We received a machine comparable in programming technique to current large computer design—a poor man's ENIAC. Only eighty decimal digits of memory, only six-by-six multiplication, and only 150 computing cycles per minute, but sequence controlled completely. Our experience indicates this program power is fundamental.

A description of specific machine features is not amiss. First, I assume you are familiar with the IBM manuals on the Type 603 Electronic Multiplier, Type 405 Accounting Machine and Type 517 Summary Punch. Just about all available extras are included.

The first addition of note is the extra connector cable. This provides for the connection from power supply to electronic unit via the accounting machine. By disconnecting the extra cable at the power supply and transferring the accounting machine end of the accounting machine-to-multiplier connection to the power supply the machines may be operated independently. This is done now and again for listing work on the accounting machine, but we rarely use the multiplier alone as few problems involve just a product of two factors with no additional calculations.

The nerve center of the combination is the two-section auxiliary control panel which has been added to the accounting machine. This panel is located just above the original three-section standard panel. The multiplier control panel located on the power supply unit is not used for combined operation. A description of the auxiliary panel and the additions to the standard panel will tell in detail what can be done with the machine.

The top six rows of the auxiliary panel require little description. The first two rows of hubs are the upper card reading brush hubs and are common to the corresponding lower panel hubs. These are used mainly for reading program X impulses for control and selection. The next two rows are the selector and chain pickup hubs and below these are two rows for the lower card reading brushes. Some program controls are wired from these brushes but mostly they are used to enter numbers into the counters or multiplier.

The truly new features of the machine begin with two rows of hubs which are counter exits. These counter exits are common to the summary punch counter exits. An extra emitter has been provided in the accounting machine parallel to the summary punch emitter to provide impulses in accounting machine time from these exits in accordance with the digit in each counter position. Also provided is the balance test impulse in the "nines" position. The use of this impulse is described below.

In the next row are column shift pickup hubs, the product exit, the product sign exit, and eight pickup hubs for eight single position selectors. The column shift arrangement requires description. It is impulsed for a specific shift position up to six places by an X impulse. The product is shifted to the right in the product exit accordingly. This saves selector wiring for varying multiplications and permits floating-decimal multiplication. The product sign is an impulse used for internally reversing the accounting machine add and subtract connections as made on the lower panel. If a counter is wired to add and a negative product sign impulse enters the counter reversal hub on the upper panel, the counter automatically subtracts. In case the product is to be used as a multiplier in the next cycle, this sign can be entered as the multiplier sign and the next product sign controlled accordingly.

Set apart by two blank rows above and below are seventeen rows of selector and chain entries. There are sixteen eight-position, four four-position and eight single-position selectors, as well as five chains each having four eight-position sets of entry hubs. Three of the entries are connected in accordance with program impulses; the fourth is connected when the chain is not impulsed. These five chains are equivalent to fifteen eight-position selectors as they are usually wired for a problem.

The remaining five rows are multiplier inputs and controls, counter reversal hubs, and counter entries. The multiplier inputs consist of two sets of six hubs for entering multiplier and multiplicand. For multiplier control there are hubs for group multiplier, half entry, multiplier and multiplicand subtract, and multiplier and multiplicand sign. When the group multiply hub is impulsed, the multiplier is retained for use with another multiplicand. Jack-plugging the half entry hubs rounds the last digit of the product in accordance with the column shift position. When the column shift switches are not wired, they can be used in lieu of selectors for accumulating partial products in over-capacity multiplications. The multiplier subtract controls are impulsed by the balance test impulse mentioned above. The left-hand position of the counter being read in is connected to the proper subtract hub. When a nine is standing in this position indicating a complement, the balance test impulse reverses the read-in circuit and the absolute value appears in the multiplier counter. At the same time the sign control relay is impulsed so that the product sign is set up accordingly. The multiplier sign hubs are used when subtractive entry is not required but the factor is negative. This happens when a product is returned to multiply or when a factor is read from a card. The X impulse just following read-in time is used for this sign. Absolute values are always used in the multiplier. The counter reversal hubs have been mentioned. In addition, the balance test impulse from a counter may be used to transfer its absolute value to another counter by impulsing the appropriate reversal hub. The counter entry hubs are common to the counter list exits on the lower panel and are just what their name indicates.

The lower panel is just about the same as the original. The principal change is the addition of thirty-two single-position selectors for control of the counter add and subtract shots. Balance test hubs for the left-hand position of each counter with exception of the two-position counters have been added. These hubs are used when operations must be discriminated by algebraic sign for lower panel wiring and are wired to immediate pickup when used for selection.

Needless to say, all work must be done in "nines complement" arithmetic. As list-out is most easily obtained through a total stroke, the progressive total device is almost always left unwired except for last card clearing. Counters are cleared during a program through card cycle total transfer.

In wiring the control panel there are two basic approaches. One is to wire for everything and program the problem on cards, the other is to wire for the specific problem. Wiring "everything" is accomplished by connecting the selectors into three chains which provide for taking amounts from any counter to the multiplier, the multiplicand, or to another counter entry. All counter entries are paralleled together in groups of eight and the counter actually entered is controlled by the add or subtract impulse. With such wiring a set of standard program instructions can be drawn up and the construction of program cards for a specific problem is mechanical. However, this is usually wasteful of time. Nearly all problems require some direct transfers of information, or multiplication by powers of ten where rounding of the final digit is not required. Such transfers may be made simultaneously with other computations provided the counter entries are not paralleled as is necessary for a standard panel. An example of this saving is shown in integrating the equation

$$\dddot{\psi} = a\ddot{\psi} + b\dot{\psi} + c(\psi - d) \ ,$$

which is related to the zeros of z in the problem above. We required only the approximate time from the initial condition

$$\psi = 0 \ , \quad \dot{\psi} = K$$

to the first zero of ψ thereafter. With some labor it could be calculated from the analytic expression

$$\psi = Ae^{-\alpha t} + Be^{-\beta t} + Ce^{-\gamma t} - A - B - C$$

but integration is much easier. Mechanical programming yields the following:

	Mult	$\dddot{\psi}$	$\ddot{\psi}$	$\dot{\psi}$	ψ	t
1	$\dddot{\psi}\Delta t$	RO				Δt
2	cd	Reset	$\dddot{\psi}\Delta t$			
3	$\ddot{\psi}\Delta t$	$-cd$	RO			
4	$a\ddot{\psi}$		RO	$\ddot{\psi}\Delta t$		
5	$\dot{\psi}\Delta t$	$a\ddot{\psi}$		RO		
6	$b\dot{\psi}$			RO	$\dot{\psi}\Delta t$	
7	$c\psi$	$b\dot{\psi}$			RO	
8		$c\psi$				

This program was used to explore the effect of varying the value of Δt. It was found that coefficient values of interest could be explored sufficiently well at $\Delta t = 0.1$. A board for the specific problem was then wired to use the following program:

	Mult	$\psi - d$	$\ddot{\psi}$	$\dot{\psi}$	ψ	t
1	$a\ddot{\psi}\Delta t$	$-d$	RO	$\psi\dot{\Delta} t$		Δt
2	$b\dot{\psi}\Delta t$	$\dot{\psi}\Delta t$	$a\ddot{\psi}\Delta t$	RO	$\dot{\psi}\Delta t$	
3	$c(\psi-d)\Delta t$	RO and reset	$b\dot{\psi}\Delta t$			
4		ψ	$c(\psi-d)\Delta t$		RO	

It is seen that four cycles suffice for an adequate computation—just half those required for the standard program setup.

Naturally all such methods necessitate a study of the accuracy required, and for rapidly changing variables the use of higher order differences rather than a smaller Δt should be considered. At this point the counter capacity may limit the possibilities. With respect to accuracy, we often save time by inquiring about the physical nature of the problem. Requests for six and eight figure accuracy are often tied to mechanisms which already rattle in the fourth significant digit.

The machine has been used almost exclusively for study of the differential equation system. However, some time has been available for other work. Wind tunnel results are computed completely at the rate of two lines of complete results per minute. The raw data are key punched, checked, and collated into a program file. The final results come out. It is hard to be enthusiastic about going back to the former chop-and-grind method between reproducer and multiplier, so the wind tunnel man waits for time on the machine when it is busy rather than start the many times longer step-by-step process. Eight by eight singular matrices iterate at a rate of one iteration in seventy seconds, a ratio of thirty to one over the older technique. Stress distribution analyses for final reports which used to take over a month can be calculated twice now in proof form, and the summary cards listed on final vellum in a day. The proofs are made to check for identical calculation. Sample calculations are made to be assured the program is correct.

Some miscellaneous items of interest to those who think in terms of individual calculations in the standard machines are twelve by twelve multiplication, long division, and extraction of square roots. For twelve by twelve multiplication the speed is about twenty products per minute if the results are punched, or twenty-five per minute for listing only. Using the desk calculator, long division can be performed and square roots can be extracted to six places in about half a minute and a minute, respectively. This is, of course, tedious. An iteration formula

$$x_{i+1} = \frac{1}{2} x_i \, (3 - N x_i^2) \ , \quad x_i \rightarrow N^{-1/2} \ ,$$

selecting automatically the first digit of the first trial reciprocal root, will converge to six places in twenty-eight card cycles. One cycle must be added for multiplication by N for $N^{1/2}$ or squaring for $1/N$ to complete the process. This gives a rate of five calculations per minute.

Future possibilities, if and when we have time, are interesting. At present we iterate matrices for solving vibrational problems. Considerable work has been done manually before we get the matrix and considerable more when we are through. The integral equations for the various vibrational modes and the ease of entering just any old function for beam characteristics appear to make iteration of the integral equation in the machine a much easier solution.

In aerodynamics our theoretical group has derived some analytic approximations to three dimensional flow in both sub- and supersonic domains. We have done a variety of computing on these. Now we feel perhaps we can just settle for the wave equation and some boundary conditions together with required mesh fineness and arrangement. Iteration will do the rest, possibly not in record time but far faster than attempting to compute from the very complicated analytic expressions.

The question of error rate should be mentioned. Naturally the machine was subject to considerable trouble at first. For this reason the initial two or three months, when we were lucky to get fifty per cent operating time, should not be included. At present we spend less than a day a week fishing for trouble. The machines separately are IBM's most dependable. The accounting machine is an old work horse of many years' proving. The Type 603 Multiplier of itself is much more dependable than the Type 601. To get back to error rate: we have operated continuously for as long as a week at a time with no errors shown in sample repetitions or manual checks of the integration problem. This is a great deal of calculating. The actual useful time available is better than eighty per cent. The principal source of lost time is in perfecting a program and in making tests and correcting wiring.

Work with the machine has also suggested some additions, not necessary but highly convenient. One such would be internal counter clearing circuits impulsed by an X. This should isolate the counter during the operation as well as make the transfer impulses available at the auxiliary panel counter exits in place of the top counter emitter impulses. Another improvement, since the eighty program card positions barely suffice, would be X-R splits between upper card brush hubs on the auxiliary and standard panels. With such an arrangement digit impulses would be available on both panels, but only X impulses on the auxiliary panel and R impulses on the standard panel. Arrangement of the auxiliary panel can be improved. We have already reversed the position of the N and X points of the selectors to permit jackplugging for chain connections but the miscellaneous single- and four-position selectors should be in the top corners with the eight-position selectors blocked solidly above the chains as well.

Increased capacity requires only more memory and selectors. With eighty computing counter positions, the memory need only store, not accumulate. Summary cards can be used, of course. Added selectors are only necessary to select from the memory. For eighty counters the selectors at present are sufficient.

DISCUSSION

[Discussion on this paper was omitted because of time limitations. A short general discussion follows Dr. Levin's paper.]

Applications of Punched Card Equipment at the Naval Proving Ground

CLINTON C. BRAMBLE

U. S. Naval Proving Ground

I WANT to make a few remarks about some of the computations that we have undertaken at the Naval Proving Ground. We are responsible to the Bureau of Ordnance for a good deal of ballistic computation work: the production of firing tables for guns, rockets, and projectiles. Our department covers other agenda, but I will speak merely of our computation work.

We have at the Proving Ground a set of the usual IBM machines, including the collator, the Type 601 and the Type 405, of which much has been said. We put in those machines as soon as IBM would furnish them to us after a conference that I had with Dr. Eckert at the Naval Observatory. I think that must have been back in 1943. We found them very useful, particularly at times when our manpower was very short, and our work load was very large. Sometimes we were operating under a contract with MIT, so that we had the output of their differential analyzer. We took the output of the differential analyzer and digested the results at the Proving Ground. A great deal of our processing was done on the IBM machinery.

Our ideal has been to produce range tables with very little being touched by human hands. We are far from that ideal at the present time. We think that in time we will approach that with our IBM machines and our Mark II Calculator as well as our battery of desk type machines. A point that I think should be made in connection with any computation laboratory is that you need a certain balance and that one type of machine does not in general render the other types unnecessary.

In addition to the machines that I have mentioned, we have one of the IBM Relay Calculators, of which five have been manufactured. Two are at Aberdeen Proving Ground, and I believe two are at the Watson Laboratory. I am not sure whether any more have been manufactured. We have just one of them, and we have put it to considerable use.

Among the various projects that we have carried out on this piece of apparatus has been the computation of sine functions to seventeen significant figures for each ten minutes of arc. This was done for a special purpose as we were about to do a job for which the most appropriate table did not seem immediately available. We have also for our own purposes calculated tables of e^x and e^{-x} to eighteen significant figures. We have also prepared density log tables for our own use, and we have done a great deal of auxiliary work in connection with production of range tables. The 601 has been particularly useful in the calculation of bombing tables inasmuch as these tables are produced by four term interpolation formulas from a general ballistic table. Among other projects that we have carried through have been the reductions of field observations of flight of bombs; we have reduced the work a good deal so that we are now in the process of eliminating graphic procedures almost entirely.

To return to the IBM Relay Calculator, we have calculated a number of solutions of a differential equation, a special case of the type that was discussed twice yesterday. Our special formula involved the calculation of the particular solution of the differential equation, and was especially concerned with the determination of a parameter in the equation so that the solution would fit a given condition.

One other piece of apparatus I should mention we have: a card-operated typewriter in which the keyboard was set up for our own special purpose. I have with me a copy of a density table which was made by the use of duplimat papers in the typewriter and directly produced from the duplimat; this is a very nice job. The heading, of course, was typed with another typewriter.

By the use of our accounting machine, range tables are also printed on specially preprinted forms. We not only print on the 405, but also use it for differencing purposes. On this density table there is a sample of six differences so produced. We can, by use of a properly set up control panel, run off the cards which contain the data on our range tables, select any two columns and get first, second, and third differences simultaneously. We use this as our check process and we find it very satisfactory. Various schemes for integrating our differential equations have been tried. I am not sure how generally these are known, but they are to be found in any books such as Bliss' *Mathematics for Exterior Ballistics*.

The differential equations of ballistics are rather troublesome; we have manipulated them many ways and introduced all sorts of independent variables for purposes of integration. We have studied integration with respect to different variables and are in the midst of exploring the possibility of doing this on the IBM Relay Calculator. I don't believe enough time studies have been made to determine whether it is going to pay off, but it is an interesting possibility.

In dealing with a network of trajectories, you first compute the slant range R and the sight angle p; that is, R and p are the polar coordinates of the projectile. These are functions of initial angle and time. Then the gunner wants the table turned inside out. He needs to know what his time of flight is going to be in terms of slant range and position angle as well as the angle of elevation for slant range and the position angle. That means that we have a tremendous amount of inverse and direct interpolation to perform. I will not bore you with the details, but the IBM equipment is very useful in that connection.

I should like to make a few more remarks which are somewhat general. In a computation section, I commented that there should be a balanced organization; that there should be a group concerned with analysis. That is the group that should have on its desks the IBM motto!

I would like to give just an example from our own work. A problem was sent to us that was concerned with projectile motions under certain specified circumstances, and it looked like a difficult problem in gyroscopic motion. It appeared on the surface to have three or four physical parameters, but by a little manipulation, the mathematician soon saw that there was only one essential parameter involved. That reduced the amount of computation from what the physicist expected to be a library when he sent it to us, to a single book. The next thing we discovered was that the motion was periodic. That meant that we could calculate merely one cycle of the motion and we had the job done. We then discovered that the motion was symmetrical with respect to a half period and that cut it down again by 50 per cent, so that in the end the amount of computation necessary was perhaps one-half of one per cent of what was contemplated by the people originally submitting the problem. That kind of analysis pays off, and I don't see how you can set up a machine and proceed to do your problems by brute strength and awkwardness. You must have them digested by an analytical group. Equations must be transformed into the most efficient system possible. Then the programming should at least be outlined by the mathematician, who breaks it down into an efficient program. It is hoped that later on coding will be entirely unnecessary, so that when the mathematician finishes his programming the job is theoretically finished.

One observation about this meeting that I would like to make, at the risk of pointing out something which you all perhaps have noticed, and that is the prevalence of problems that are common to so many different interests. Dr. King, in speaking of his problems in connection with physical chemistry, finally came around to mentioning such things as statistics.

I should like to go back and say that, when I was a graduate student and took my degree, I came through with very considerable mathematical purity; by force of circumstances in the first World War, I was driven into applied mathematics, and to my own surprise, I found it just as fascinating as pure mathematics. In fact, I was weaned away from pure mathematics, because you can't do too many things in one lifetime and do them all well. This experience has been duplicated by many people forced into applied mathematics in the last war who have found that the problems they got into were just as intriguing, offered just as much challenge, were just as hard to do, and in fact, had something about them that is not generally suspected. If you go into this computation game sufficiently, just when you feel that you are getting away from pure mathematics, you find that the problems have led you into analysis and have forced you right back to pure mathematics again. I think that no young man who is going into scientific work at the present time should have any idea that he is misusing his talents in any way by going into this particular field.

I want to say in conclusion, I appreciate very much being a member of this group and this meeting. I have met some old friends here, and I have become acquainted with many people whom I knew by name only and whose acquaintance I hope to keep in the future.

DISCUSSION

[Discussion of this paper was omitted because of time limitations.]

Use of the IBM Relay Calculators for Technical Calculations at Aberdeen Proving Ground

JOSEPH H. LEVIN

Aberdeen Proving Ground

THE TWO IBM Pluggable Sequence Relay Calculators at the Ballistic Research Laboratories, Aberdeen Proving Ground, are only part of a battery of computing facilities. In addition to these machines and an assortment of standard IBM equipment, the Laboratories have at the present time a differential analyzer, the ENIAC (Electronic Numerical Integrator and Computer), and a Bell Relay Computer. In an installation of this sort the first decision to be made when a new problem comes in is not how to program it, but where to assign it. This decision is largely a question of economics. The relay calculators have neither the speed of the ENIAC nor the versatility of the Bell computer. For programs of lesser or intermediate difficulty the IBM relay calculators are highly valuable. For long and intricate programs use of the Bell machine or the ENIAC would be indicated. To assign a simple calculation involving only a few operations to either of these machines would clearly be a waste of talent. Of course, the decision as to where a problem is to be assigned must also be controlled by such other considerations as urgency of the problem, current work loads, and so on.

As is generally true of machine calculations, in order that a problem be adaptable to the relay calculators, it must be reducible to a routine. For the problems handled on the Aberdeen relay calculators, the number of arithmetic operations involved per routine has ranged from perhaps five or six in the simplest cases to somewhere in the neighborhood of a hundred in the most complex. The latter are probably beyond the capacity that the machines were originally designed to accommodate.

Inasmuch as the computing facilities at the Ballistic Research Laboratories are available to outside agencies for work on government scientific projects, it would be well for such agencies to have some idea of the kinds of problems that have been handled on the various types of equipment. The following exposition, confined to the IBM Relay Calculators, is not intended to be a description of the machines themselves, but is a brief account of some types of problems carried out on them. However, there are several facts about the machines that should be pointed out before launching into a discussion of their applications. The Relay Calculator has two feeds, a reproducing feed and a punching feed. Cards may be read from both feeds, but can be punched only in the punching feed. Cards are fed in each feed at a maximum rate of one hundred cards per minute. This rate, however, depends upon the complexity of the routine. For purposes of reading in constants there are four groups of ten-way switches. The machine may be programmed to perform routines of a fair degree of complexity consisting of any of the operations of addition, subtraction, multiplication, division, or square root. More extensive information on the characteristics and principles of operation of this machine is available in MTAC,[1] and also in a forthcoming publication of the Ballistic Research Laboratories.[2]

One rather frequent type of problem is that of subtabulating or interpolating in existing tables. For a third order interpolation the calculation may be performed as follows:

$$F_n = F_o + n\,[A + n(B + Cn)]\ ,$$

where n is the fractional part of the interval, and A, B, C are functions of the tabular differences. A, B, C are precomputed and punched on the cards, while n may be read from the switches or from cards in either the punching feed or reproducing feed. The whole operation is easily programmed for a single run of the cards.

An extensive series of calculations has involved the determination of a great number of direction cosines from point coordinates. There is nothing remarkable about this accomplishment. However, it is worth pointing out that it is a calculation which would be wasteful of the capabilities of the ENIAC or the Bell machine. At the same time, it is an awkward calculation on the standard IBM equipment. On the other hand, this calculation is ideal for the relay calculator, being easily programmed and rapidly accomplished, and it does not tax the facilities of the machine. Another rather extensive series of calculations of the same order of simplicity has been a series of velocity and acceleration determinations, given the coordinates of successive points on a trajectory as a function of time. Again it would be wasteful for this work to be carried out on the ENIAC or the Bell machine. But the problem is ideally suited to the IBM Relay Calculator.

I shall describe the following problem in somewhat greater detail because of its interest both from the theoretical and computational points of view. We are given the following partial differential equation together with the indicated boundary conditions:

$$\partial v/\partial t = \partial^2 v/\partial x^2 + m\mu(t)\,\partial v/\partial x\,, \quad (x \geqq 0\,,\quad m = \text{const.}) \tag{1}$$

$$v(x,t) \to 0 \quad \text{as} \quad x \to \infty \tag{2}$$

$$-2\partial v(0,t)/\partial x = 1 - \mu(t) \tag{3}$$

$$v(0,t) = 1/\sqrt{\pi} \tag{4}$$

$$v(x,0) = e^{-x^2/4}/\sqrt{\pi} - \frac{1}{2}x\left(1 - 2/\sqrt{\pi}\int_0^{x/2} e^{-a^2}\,da\right). \tag{5}$$

It is desired to obtain solutions of (1) corresponding to assigned values of the constant m and subject to the boundary conditions (2) through (4). The usual numerical procedure consists of replacing (1) by a difference equation and finding the solution of this equation as an approximation to that of the partial differential equation. Putting

$$x = i\Delta x,\quad t = j\Delta t \qquad (i,j = 0, 1, 2, \ldots)$$
$$v(i,j) \equiv v(i\Delta x, j\Delta t),\quad \mu_j \equiv \mu(j\Delta t),$$

and replacing the partial derivatives in (1) by finite difference approximations, the following difference equation is obtained:

$$\frac{v(i,j+1) - v(i,j)}{\Delta t} = \frac{v(i-1,j) - 2v(i,j) + v(i+1,j)}{(\Delta x)^2} + m\mu_j\frac{v(i+1,j) - v(i-1,j)}{2\Delta x}\,.$$

Letting $b \equiv \Delta t/(\Delta x)^2 = \frac{1}{2}$, this equation may be written

$$v(i,j+1) = \frac{1}{2}\left(1 - \frac{1}{2}m\Delta x\mu_j\right)v(i-1,j) + \frac{1}{2}\left(1 + \frac{1}{2}m\Delta x\mu_j\right)v(i+1,j)\,. \tag{6}$$

We also substitute a finite difference approximation for the boundary conditions (3). Using Newton's forward interpolation formula we have

$$\partial v_o(j)/\partial x = 1/\Delta x\,(\Delta v_o - 1/2\Delta^2 v + 1/3\Delta^3 v + \ldots),$$

where $v_o(j) \equiv v(0,j)$. For purposes of the present illustration we shall not go beyond third differences. Writing the differences in terms of the functional values and substituting in (3), we obtain the following approximate formula for μ_j:

$$3\Delta x\,\mu_j = 3\Delta x - 11/\sqrt{\pi} + [18v(1,j) - 9v(2,j) + 6v(3,j)] \tag{7}$$

The solution of (6) with boundary conditions (7), (2), (4), and (5) leads to a function $v(i,j)$ defined over a rectangular network of points. The procedure for solving consists in starting with the function (5) defined for the row $j = 0$, and with $\mu_0 = 0$, and proceeding by means of (6) to the row $j = 1$. The quantity μ_1 is then determined by (7) and is used in (6) together with the boundary condition (4) to determine v for $j = 2$, etc. A check on the procedure is derived in the following manner: both sides of (1) are integrated with respect to x between the limits 0 and ∞, giving

$$\int_0^\infty v_t(x,t)\,dx = \int_0^\infty [v_{xx}(x,t) + m\mu(t)v_x(x,t)]dx\,.$$

In view of the boundary conditions (2), (4), and (3), this may be written

$$\int_0^\infty v_t(x,t)\,dx = \frac{1}{2}\left[1 - \frac{\sqrt{\pi} + 2m}{\sqrt{\pi}}\mu(t)\right].$$

That is,

$$\mu = \mu_e - 2\mu_e\int_0^\infty v_t(x,t)\,dx\,, \tag{8}$$

where

$$\mu_e = \frac{\sqrt{\pi}}{\sqrt{\pi} + 2m}$$

is the steady state value of μ. Integrating the two sides of equation (8) between the limits 0 and t, interchanging the order of integration on the right, and making use of the fact that

$$\int_0^\infty v(x,0)\,dx = \frac{1}{2}\,,$$

we obtain

$$\sigma(t) \equiv \int_0^t \mu(t)\,dt = \mu_e\left[(t+1) - 2\int_0^\infty v(x,t)\,dx\right].$$

By computing $\sigma(t)$ in the two ways shown in the foregoing equation, a numerical check is obtained.

The problem described is well suited to a machine of the type of the IBM Relay Calculator. The machine method conforms to the hand computation procedure in that a row of points is computed at a run. For any row j = constant, the input cards in the reproducing feed consist of cards punched with the values $v(i,j)$. In the first four of these cards are punched the constants appearing in (7) : $3\triangle x - 11/\sqrt{\pi}$, 18, −9, and 6, respectively. At the beginning of each run the constants

$$\frac{1}{2}\left(1 - \frac{1}{2}m\triangle x\mu_j\right), \frac{1}{2}\left(1 + \frac{1}{2}m\triangle x\mu_j\right) \qquad (9)$$

are set in two rows of switches. A stack of blank cards is fed into the punch feed. As the input data are fed through the reproducing feed, the machine uses the values $v(i,j)$ to compute the values $v(i,j+1)$ for the next row by (6), and punches them into the cards going through the punching feed. Using the first four values $v(0,0)$, $v(1,0)$, $v(2,0)$, and $v(3,0)$ the machine also computes $3\mu_1\triangle x$ by (7), and the constants (9) to be used in the next run. At the end of the run the output cards are removed from the punch stacker, the constants (9) are read and set in the switches, the cards are placed in the reproducing feed while another deck of blank cards is placed in the punching feed, and the process is repeated.

The interval $\triangle t$ must be chosen small to begin with, for $\mu(t)$ behaves like $\sqrt{t}$ for small t, so that $\mu'(0)$ is infinite. As the solution progresses and the steady state condition is approached, $\triangle t$ can and should be increased to keep the computation time within reasonable bounds. It must be remembered that in order to satisfy the condition $b = 1/2$, $\triangle x$ must be changed whenever $\triangle t$ is. In the problems carried out on the relay calculators, the interval $\triangle x$ was doubled whenever it was found that the results obtained upon using the larger interval were the same as those obtained by use of the smaller interval. Solutions were carried out for the cases m = 0, 1, 2, 5, and 10.

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DISCUSSION

Dr. Herget: I have found it pretty dangerous to use $b = 1/2$. Haven't there been papers showing that the solution may not be stable for the value one-half?

Dr. Levin: The equation we are dealing with is of parabolic type. It may be shown that for such an equation the condition for stability is $\triangle t/(\triangle x)^2 \leqq 1/2$. Also a big reason for using $b = 1/2$ is that it yields a simpler formula.

Dr. Thomas: May I take two minutes for a statement about this sort of problem?

Suppose you have fixed ranges and even intervals in x. What you are essentially doing in this sort of work in going to finite differences is exactly equivalent to using a finite number of terms in a Fourier expansion. You have terms, for instance, in sin x, sin $2x$, sin $3x$, . . . ; these in the exact solution should decay with time at rates e^{-t}, e^{-4t}, e^{-9t}, . . . The difference formulas, besides only giving a finite number of terms of the expansion, make the terms given decay at the wrong rates.

The later terms decay at less and less accurate rates. In the simplest case, you can show that if you have n points corresponding to n terms of the expansion, the first few terms will decay nicely; but as you come to higher terms, the end terms decay more and more slowly, and it is usually the case that the last term you keep decays a little less rapidly than the first. So all you can do is get the limiting form; you get nothing exact about the details after some time.

You have to look at this sort of thing fairly closely to estimate what is happening, especially if you hope to extend it to more complicated cases with curved boundaries, and so on.

GENERAL DISCUSSION

Dr. Eckert: We have a few minutes to wind up the Endicott part of our conference. With reference to the last papers on special equipment, I might say that you will see two relay calculators of the type Dr. Levin and Dr. Bramble mentioned, and a prototype combination machine somewhat like Dr. Fenn's, at the Watson Laboratory. The card-operated typewriter at the SSEC is also on display.

Are there other comments or questions?

Mr. Kintas: We have been considering solving natural frequency problems for free crank systems and compound beams. Mr. Mack suggested that both problems could probably be handled by a single setup of matrix equations using the coefficients and unknown terms which describe the conditions of kinetic and potential energy in the system. Other aircraft men here have investigated that problem.

Mr. Harman: What is the maximum capacity of the Type 602 in multiplying square matrices of the nth order? Suppose the elements are two-digit numbers; how many can be handled at once by the 602?

Dr. Bramble: If you put one element on a card there is no limitation on n.

Dr. Grosch: But then multiplying two unsymmetrical matrices together would require $2n^3$ cards to pass through the 602. If there were an unlimited number of program steps available, one could store perhaps thirty-two

two-digit numbers in the machine, reserving only counters 1, 3, 6, 9 and 10 for operational use. I am assuming all elements are positive. Then every other card passing into the 602 would carry up to sixteen elements of matrix A (one row) and the next would carry up to sixteen elements of B (one column); one element of A·B would be accumulated in LHC 9 and 10 and punched on the B cards. Only $2n^2$ cards would pass into the 602, for $n \leqq 16$. But since fifteen recalculation cycles would be required, the standard machine would be limited to smaller values of n. And many selectors would be needed, especially if negative elements did occur.

I would like to make a statement about the bibliography on technical applications of punched card methods we put out at the Watson Laboratory in 1947.[1] If any of you have not seen it, you can secure copies through the IBM Department of Education in Endicott. We want to correct, improve and expand that first attempt. In the foreword we requested suggestions as to format, and especially additions to the list of references. I am sorry to have to report that in the first year not one single item has come in except through our own efforts. If some of you will help, we might revise and expand the bibliography annually.

REFERENCE

1. International Business Machines Corporation, "Bibliography [on] the Use of IBM Machines in Scientific Research, Statistics, and Education," Form 50-3813 (1947).

Simultaneous Linear Equations

FRANCIS J. MURRAY

Columbia University

I SHOULD LIKE to discuss linear equations and the solution of linear equations more or less in connection with the type of machines that we have developed. I am sure that you are all familiar with many other approaches, and the mathematical basis of what I would like to say is reasonably well known. The usual theory of linear dependence such as one finds in Bôcher, and the relatively simple vector constructions, are to be considered not absolutely but to within a certain accuracy. It seems to me that this has to be done as soon as one considers systems in which the size of the results is not obvious.

Now let me briefly describe our objectives in building the machine which is now in use at the Watson Laboratory. We set up a machine to solve a system of equations

$$\Sigma_j\, a_{ij}\, x_j = b_i \ . \qquad (1)$$

However at all times the device regards the x_j's as well as the a_{ij}'s and b_j's as inputs and evaluates by relatively simple means

$$\mu = \Sigma_i\, (\Sigma_j a_{ij} x_j - b_i)^2 \ , \qquad (2)$$

which it represents on a meter. One finds a solution of the equation by manipulating the x_j's so as to minimize μ. This manipulation process is always convergent, an advantage enjoyed by no other adjusting device. If there is a solution, we obtain it.

Another objective was to put in the coefficients digitally, and we succeeded in doing this. The use of alternating current raised certain phase difficulties and these Mr. Walker settled.[1]

In attacking a system of equations (1), we begin by rounding to three figures. Normally any accuracy can be obtained by an iterative process which involves merely a change of the constants b_i at each step. However, this is clearly ineffective when dealing with a problem of this sort:

$$\begin{aligned} 1.000000\,X + 0.999000\,Y &= 1.000000 \\ 0.999000\,X + 1.000000\,Y &= 0.000000 \ , \end{aligned}$$

since the problem is indeterminate relative to the first three digits. Nevertheless our machine can be effectively used here. Before discussing this method, let us point out that the six figure accuracy of the coefficients is only adequate to give about three figures in the result. Eliminating Y yields an equation in X

$$0.001999\,X = 1.000000$$

where, due to rounding, the last 9 in the coefficient of X may be off by more than one. The first three digits for which the matrix of coefficients is singular also corresponds to a loss of accuracy in the answer.

We have a reasonably routine method of handling problems of this type on our three-digit machine. Mathematically this process consists in looking for a linear dependence among the columns of the coefficient matrix and by a suitable change of variables eliminating them. Notice that if in our example we let $X = (x + y)$ and $Y = x - y$ our equations become

$$\begin{aligned} 1.999000\,x + 0.001000\,y &= 1.000000 \\ 1.999000\,x - 0.001000\,y &= 0.000000 \ . \end{aligned}$$

By a change of y scale, i.e., $y = 1000\,z$ this becomes

$$\begin{aligned} 1.999\,x + 1.000\,z &= 1.000 \\ 1.000\,x - 1.000\,z &= 0.000 \end{aligned}$$

This problem can be readily solved on the machine. The coefficient of z now has only three significant digits but these are all usable in the machine and we can obtain all the accuracy the problem justified.

The advantage of this type of device is that the essential linear combinations can be found on the machine. Consider our original system of equations

$$\begin{aligned} a_{1\ 1}\,x_1 + \ldots + a_{1\ 12}\,x_{12} &= b_1\, t \\ \ldots \quad \ldots \quad \ldots \quad &\ \ldots \\ a_{12\ 1}\,x_1 + \ldots + a_{12\ 12}\,x_{12} &= b_{12} t \ . \end{aligned}$$

In the machine, the b_i's appear multiplied by a gauge variable t. This has the result that the unknowns x_i are in the form x_i/t and hence not necessarily between -1 and $+1$. Suppose that the third column is very nearly a linear combination of the remaining columns. We can readily find this on the machine as follows: We set $t = 0$, $x_3 = 1$ and minimize

$$\mu = \Sigma_i\, (a_{i1}\,x_1 + a_{i2}\,x_2 + a_{i3} + \ldots + a_{i12}\,x_{12})^2$$

relative to $x_1, x_2, x_4, \ldots, x_{12}$. This yields the coefficients

$y_1, y_2, \ldots, y_{12}$, of the linear combination and we can use these to compute digitally the terms

$$a_{i3} = a_{i1}\, y_1 + a_{i2}\, y_2 + a_{i3} + a_{i4}\, y_4 + \ldots + a_{i12}\, y_{12} \;.$$

In this computation all the digits of the a_i's are used. We then can rewrite our original system of equations

$$\begin{aligned} a_{1\,1}\,(x_1 - y_1x_3) + a_{1\,2}\,(x_2 - y_2x_3) + a_{1\,3}\,x_3 + \ldots \\ + a_{1\,12}\,(x_{12} - y_{12}x_3) = b_1t \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{12\,1}\,(x_1 - y_1x_3) + a_{12\,2}\,(x_2 - y_2x_3) + a_{12\,3}\,x_3 + \ldots \\ + a_{12\,12}\,(x_{12} - y_{12}x_3) = b_{12}t \;. \end{aligned}$$

We now have a system of twelve equations in the twelve unknowns

$$z_1 = x_1 - y_1x_3\,,\; z_2 = x_2 - y_2x_3\,,\; z_3 = x_3\,, \ldots, \\ z_{12} = x_{12} - y_{12}\,x_3 \;.$$

The a column has been obtained by a minimizing process relative to the other columns, and if this has been carried out completely the a column will be orthogonal to the other columns. The a column will have smaller quantities in it than the a_{ij} and consequently fewer digits. In these circumstances, however, we can always change the scale of x_3 and, in the machine, utilize the values of a_{i3} to within 0.1 per cent of the largest a_{i3}. The loss of accuracy represented by the fewer digits in the other a's corresponds precisely to loss of accuracy in these equations themselves, i.e., the loss of accuracy involved in passing from coefficients to unknowns. On the other hand, we have eliminated a linear dependence among the coefficient columns and thus permitted the solution process to proceed.

When there is only one approximate linear dependence among the columns, the above process permits us to eliminate it by modifying just one column of the coefficient matrix, i.e., just changing one card. If there are a number of these dependencies, a number of these steps must be taken. Theoretically, this modification process can be pushed to an extreme. We begin by minimizing the second column relative to the first. This yields a second column orthogonal to the first. Then we can minimize the third column relative to these two. This process can be repeated until we have a matrix whose columns are orthogonal, and this can be inverted immediately with practically no loss of accuracy. Let A be the original matrix, T the new orthogonal matrix. We have found a matrix C such that $AC = T$ and hence $A^{-1} = CT^{-1}$. Since the columns of T are mutually orthogonal, the inverse T^{-1} of T is very simply connected with its transpose. Thus if T is the matrix $\{t_{ij}\}$ where i is the row index, j the column index and $\tau_j = \sum_i t_{ij}^2$, then T^{-1} is simply $\{t_{j,i}/\tau_j\}$. Note that the accuracy of T^{-1} is essentially that of T and the loss of accuracy for the system is represented by the change of scale factors in C.

These difficulties are present of course in every method of solving simultaneous linear equations since we are always limited in the number of digits available. Even in the elimination process one must always keep in mind the retention of maximum accuracy. For instance, the von Neumann-Goldstine estimate of the loss of accuracy involved in the elimination process is based on the assumption that the largest coefficient available is used as a divisor in each step.[2] Failure to follow this as a policy can lead to a serious loss of accuracy. If we merely choose a coefficient which is half as big as the largest, in the twelfth order case at each step we will have lost unnecessarily an accuracy factor of $2^{11} = 2048$. Pyramids have been built on factors like this.

Finally, one should point out that the type of quasi-singularity I have discussed above is by no means uncommon. In the majority of practical problems, full digital regularity is exceedingly difficult to obtain. Instead some aspect of high accuracy may be reasonably available. For instance, to locate a point in a plane by linear observation, two observation points far apart could be used, lines from each of these points drawn, and their intersection found. This would yield full digital regularity in the corresponding linear equation problem. However, this is not always practical. Instead one may have to take the observation points near together and very accurately measure the direction angles of the lines.

This situation generalizes. Full digital regularity is frequently nonobtainable and we must substitute high accuracy procedures instead. This leads to these quasi-singular situations.

REFERENCES

1. R. M. WALKER, "An Analogue Computer for the Solution of Linear Simultaneous Equations," *IRE Proc.*, 37 (1949), pp. 1467-73.
2. VON NEUMANN and GOLDSTINE, *op. cit.*, pp. 1051-52.

DISCUSSION

Dr. Blanch: Are you publishing this, Dr. Murray?

Dr. Murray: First I am going to write an operational manual for the machine; that will be available, anyway. Then I plan to publish a general consideration of the rate of convergence of the whole process.

Mrs. Rhodes: Will it be something that will be helpful? You know what we go through; we get the highest and lowest characteristic values and then divide them to get the upper bound of the error. We have to know how close the determinant is to zero. I hope you will give us something simple so we can go ahead and make use of it.

Dr. Murray: I have been trying to get something as simple as possible. I know what you mean; our interest dies, and we do not determine the least characteristic root.

Computation of Shock Wave Refraction on the Selective Sequence Electronic Calculator

HARRY POLACHEK

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TECHNOLOGICAL developments in a number of scientific fields have reached the stage where the important bottleneck at the present time is not the difficulty in formulating the mathematical equations involved, but rather in obtaining numerical results from these equations which are applicable to specific problems. An outstanding illustration is the field of aerodynamics. The equations of motion for aerodynamic problems have been derived and formulated many years ago, and account has been taken of various effects such as compressibility, viscosity and heat conductivity. The equations have also been written in three dimensions, as well as in two or one dimension.

However, in applying these equations it is found that only a few cases can actually be solved numerically; and these special cases are usually not applicable to the phenomena that actually take place. For instance, compressibility is usually left out, and so is viscosity and thermal conductivity, while equations are only solved for one or two dimensions because solutions in three dimensions are too difficult.

I would like to cite another example (a problem which I had to handle during the war) which illustrates the difficulty of obtaining numerical results from mathematical equations. This problem arose in connection with the question of heat sensitivity of an explosive. The question was: If a portion of an explosive is heated to a certain temperature, will it explode, or will the heat be dissipated harmlessly and no explosion take place?

The equation describing this problem is just a slight variation of the ordinary heat equation for which a solution exists and can be obtained numerically in a matter of minutes. But there was an extra term involved which rendered the equation nonlinear. With the addition of that term the solution was so difficult that without access to automatic machines we had to spend almost a year to find the solution for just one single set of boundary conditions.

The type of problems that are most difficult, and probably those which will require the greatest use of modern calculating machines in the future, are problems involving partial differential equations. However, even in the case of systems of algebraic equations, the difficulties may be so great that standard equipment cannot be relied upon to give sufficient data to enable the scientist to survey the problem. As a matter of fact, the problem which I am going to discuss today is a problem which may be represented mathematically as a system of simultaneous algebraic equations. It is the problem of the refraction of a shock-wave at a free surface separating two gases.

The problem is illustrated in Figure 1. There are two media of different physical characteristics. A shock-wave originating in one of these media is propagated toward the interface which separates the two media, and strikes the interface at a given angle ω.

By analogy with the case of normal incidence it may be expected that a shock-wave will be transmitted into the second medium, while either a shock-wave or a rarefaction wave will be reflected into the original medium. In the first

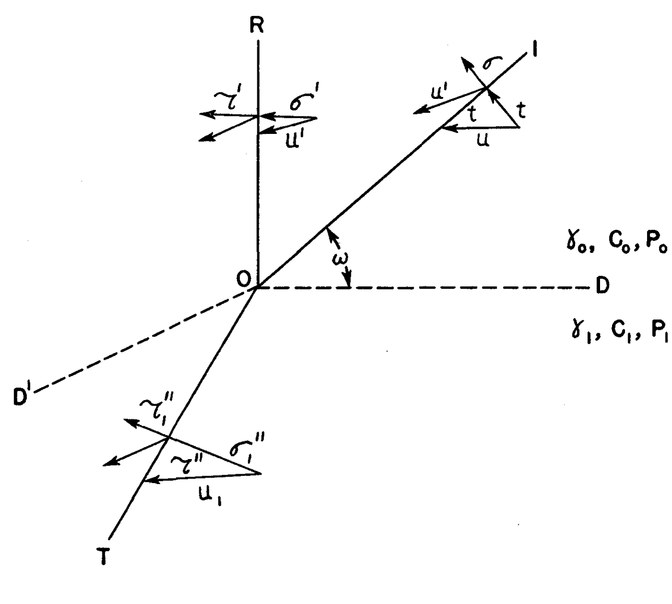

FIGURE 1. TRIPLE SHOCK CONFIGURATION

instance, the resulting pattern will be a triple-shock configuration; in the second case a refraction configuration will be formed consisting of two confluent shocks and an angular rarefaction wave.

The physical formulation of the problem is the following: we have to satisfy the well-known Rankine-Hugoniot relations across the shock-waves. There are three shock-waves in all. (In the case where a rarefaction wave rather than a shock-wave is reflected, we must replace the Rankine-Hugoniot relations with those governing a Prandtl-Meyer angular rarefaction wave.) In addition, we must satisfy the condition of equality of pressures on both sides of the interface, and the condition that the flow on both sides of the interface must be parallel, not necessarily of the same magnitude.

If we combine all these conditions we obtain the system of equations which is exhibited below.

Given: $\xi, \omega, \gamma, \gamma_0, \dfrac{c}{c_0}$

A. Compute

$$\sigma^2 = \frac{(\gamma+1)\xi+(\gamma-1)}{2\gamma}, \quad \tau = \frac{(\gamma-1)\sigma^2+2}{(\gamma+1)\sigma}, \quad \eta = \frac{(\gamma+1)\xi+(\gamma-1)}{(\gamma-1)\xi+(\gamma+1)} \tag{1}$$

$$t = \tau \cot \omega \tag{2}$$

$$h = \frac{\tau\sqrt{\eta}}{\sin \omega}\left(\frac{c}{c_0}\right) \tag{3}$$

$$M_1 = \sqrt{\sigma^2+t^2}, \quad \theta_1 = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{\gamma-1}{\gamma+1}(M_1^2-1)}\right] \tag{4}$$

B. Find k for which $F(k) = 0$

$$F(k) \equiv [b'(ac'+a'c)+b(a'c'-ac)][dfc'-egc] - [b'(a'c'-ac)-b(ac'+a'c)][efc'+dgc] \tag{10}$$

where $a = \sin \omega$, $a' = \cos \omega$, and

$$\theta_2 = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{\gamma-1}{\gamma+1}(k^2M_1^2-1)}\right] \tag{5}$$

$$b = \frac{1}{kM_1^2}\left[\left(1+\sqrt{(M_1^2-1)(k^2M_1^2-1)}\right)\cdot \sin(\theta_2-\theta_1) - \left(\sqrt{k^2M_1^2-1}-\sqrt{M_1^2-1}\right)\cdot\cos(\theta_2-\theta_1)\right]$$

$$b' = \frac{1}{kM^2}\left[\left(1+\sqrt{(M_1^2-1)(k^2M_1^2-1)}\right)\cdot \cos(\theta_2-\theta_1) + \left(\sqrt{k^2M_1^2-1}-\sqrt{M_1-1}\right)\cdot\sin(\theta_2-\theta_1)\right]$$

$$d = t, \quad e = \sigma \tag{6}$$

$$\xi' = \left[\frac{(\gamma-1)M_1^2+2}{(\gamma-1)k^2M_1^2+2}\right]^{\frac{\gamma}{\gamma-1}} \tag{7}$$

$$f^2 = \frac{(\gamma_0+1)\xi'+(\gamma_0-1)\xi}{2\gamma_0}, \quad g = \frac{(\gamma_0-1)f^2+2\xi}{(\gamma_0+1)f} \tag{8}$$

$$c = \frac{f}{h}, \quad c' = -\sqrt{1-c^2} \tag{9}$$

This is a rather complicated system of equations. To begin with, there is a total of five independent parameters that enter the problem: The strength of the shock-wave ξ, the angle ω between the incident shock-wave and the interface, and three other constants which characterize the gases. Secondly, the unknown quantities are given in implicit, rather than explicit, form. Starting with the basic parameters, we are able to calculate directly all quantities entering in equations (1) to (4). We then must satisfy equation (10), which is the basic equation, and which represents the equality of direction of flow on both sides of the interface. We must find the value of k for which $F = 0$. However, the quantities which express the function F themselves involve the letter k.

We were obliged to proceed as follows: We chose k equal to unity, and a second value obtained from the preceding problem. For these two values we calculated the function F. We then determined if these were of opposite sign. If they were, we were certain that the solution was confined within these two limits; and we proceeded by interpolation to obtain a more accurate solution. If the two values of F were not of opposite sign we could not ordinarily apply interpolation, because the process may not converge. In that case we chose a different value of k in such a manner as to produce two values of F of opposite sign.

Figure 2 is a compact chart giving the sequence of operations on the machine, the coding for which was carried out by the staff of the laboratory here, particularly by Mr. Skillman and Miss Hanson under the direction of Mr. Clark. The main sequence was used to calculate the value of F for $k = 1$, and to test whether (for this value of k) $F = 0$. If this was the case, the machine proceeded to sequence K, otherwise to sequence L, which was used to compute the value of the function F for the value of k of the preceding problem. The machine then examined the

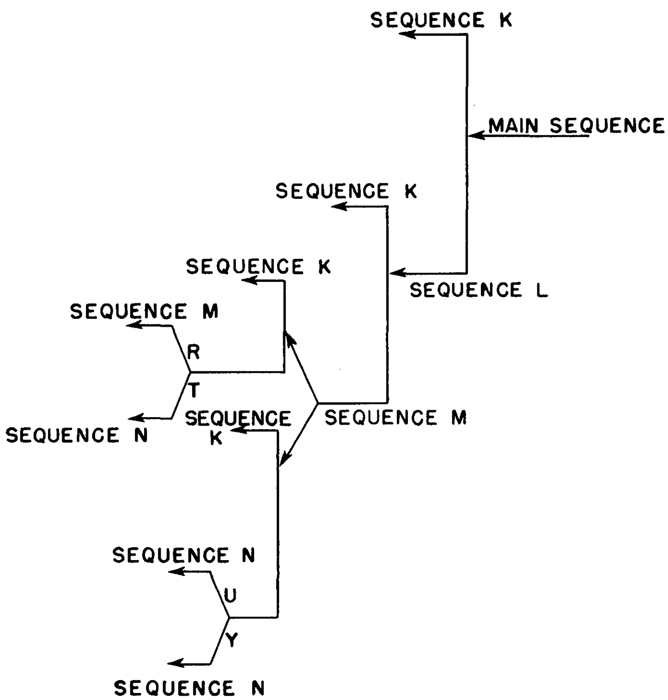

FIGURE 2. PROGRAM CONTROL CHART

two values of F and determined if they were of opposite sign or not. If they were, the machine was instructed to go to sequence N. If they were not, the machine proceeded to sequence M, which was used to obtain a new value for k and a corresponding value for F. Eventually, when opposite signs were obtained, the machine proceeded to sequence N. It then remained on sequence N, interpolating between the two last values of k, until the final value of F equal to zero was obtained.

Before proceeding with the actual analysis of the solutions which we obtained, I would like to point out that a considerable amount of analysis had to be carried out prior to the numerical calculations on the machine. In addition to the complexity of the system of algebraic equations involved, a number of other factors had to be given careful consideration. First, the system of equations involved possesses a large number of extraneous mathematical solutions. It may be shown that it is equivalent to a twelfth order polynomial equation, and thus may have a maximum of twelve roots. Some of these are real and some complex. Usually there were multiple solutions, and we had to determine beforehand which of these were physically plausible. We accomplished this by tying up our solutions with the previously known solutions, for an acoustic wave (for which Snell's law of refraction holds) and for a wave at normal incidence (for which a unique solution exists).

Another difficulty is that there are two types of refraction patterns which may occur. One is a refraction pattern with a reflected shock-wave, and the second is a refraction pattern with a reflected rarefaction wave. We had to be able to tell *a priori* which one of these two phenomena will occur. We were able to do that by connecting any solution with that of normal incidence and by introducing a so-called transition angle, at which point transition takes place from one type of refraction pattern to the other.

TABLE I

Problem	γ	γ_0	$(c_0/c_1)^2$	Gases
1	5/3	1.4	0.835	Argon-Nitrogen
2	1.4	5/3	0.120	Air-Helium
3	1.4	1.4	0.875	Oxygen-Nitrogen
4	1.4	4/3	0.600	Air-Methane
5	4/3	1.4	0.600	CarbonDioxide-Air
6	1.1	1.4	0.190	Freon-Air
7	1.1	5/3	0.020	Freon-Helium
8	5/3	1.1	0.800	Krypton-Propane
9	5/3	5/3	0.240	Krypton-Neon
10	1.1	5/3	0.460	Freon-Krypton
11	1.1	5/3	0.600	Propane-Argon
12	1.1	1.4	0.013	Freon-Hydrogen
13	1.4	5/3	1/.835	Nitrogen-Argon
14	5/3	1.4	1/.120	Helium-Air
15	1.4	1.4	1/.875	Nitrogen-Oxygen
16	4/3	1.4	1/.600	Methane-Air
17	1.4	4/3	1/.600	Air-CarbonDioxide
18	1.4	1.1	1/.190	Air-Freon
19	5/3	1.1	1/.020	Helium-Freon
20	1.1	5/3	1/.800	Propane-Krypton
21	5/3	5/3	1/.240	Neon-Krypton
22	5/3	1.1	1/.460	Krypton-Freon
23	5/3	1.1	1/.600	Argon-Propane
24	1.4	1.1	1/.013	Hydrogen-Freon

Table I gives a complete list of all problems considered. There were a total of twenty-four problems which involve twelve different gas combinations. We chose these particular gas combinations for several reasons. First, these are typical of the gases that will be used in any experimental work in the future; secondly, they possess properties which exhibit the various types of solutions which may occur. We wanted to obtain solutions for which the refraction pattern is of the shock-wave variety for normal incidence, and which go over, at the transition angle, to the opposite type. We wanted also to consider cases for which no transition takes place. We also wanted to include problems for which normal incidence produces a reflected rarefaction wave rather than a shock-wave. We will discuss several of these cases in some detail.

Problems 3 and 15 are both concerned with the same gas combination (oxygen and nitrogen). These have the same values for the gas constants (ratio of specific heats 1.4), while the ratio of velocity of sound for the two is 0.875.

We consider Problems 3 and 15 together for the reason given above. For both Problems 3 and 15, as may be seen from Table II, the transition angle ω_t exists for all strengths ξ of the incident shock. In Problem 15, however, the limiting angle ω_1 (for which the material speed behind the incident shock becomes sonic) precedes the transition angle ω_t (i.e., $\omega_1 < \omega_t$ for strong shocks, viz., $\xi = 0.0$ and 0.1). Thus no transition can take place for these strengths.

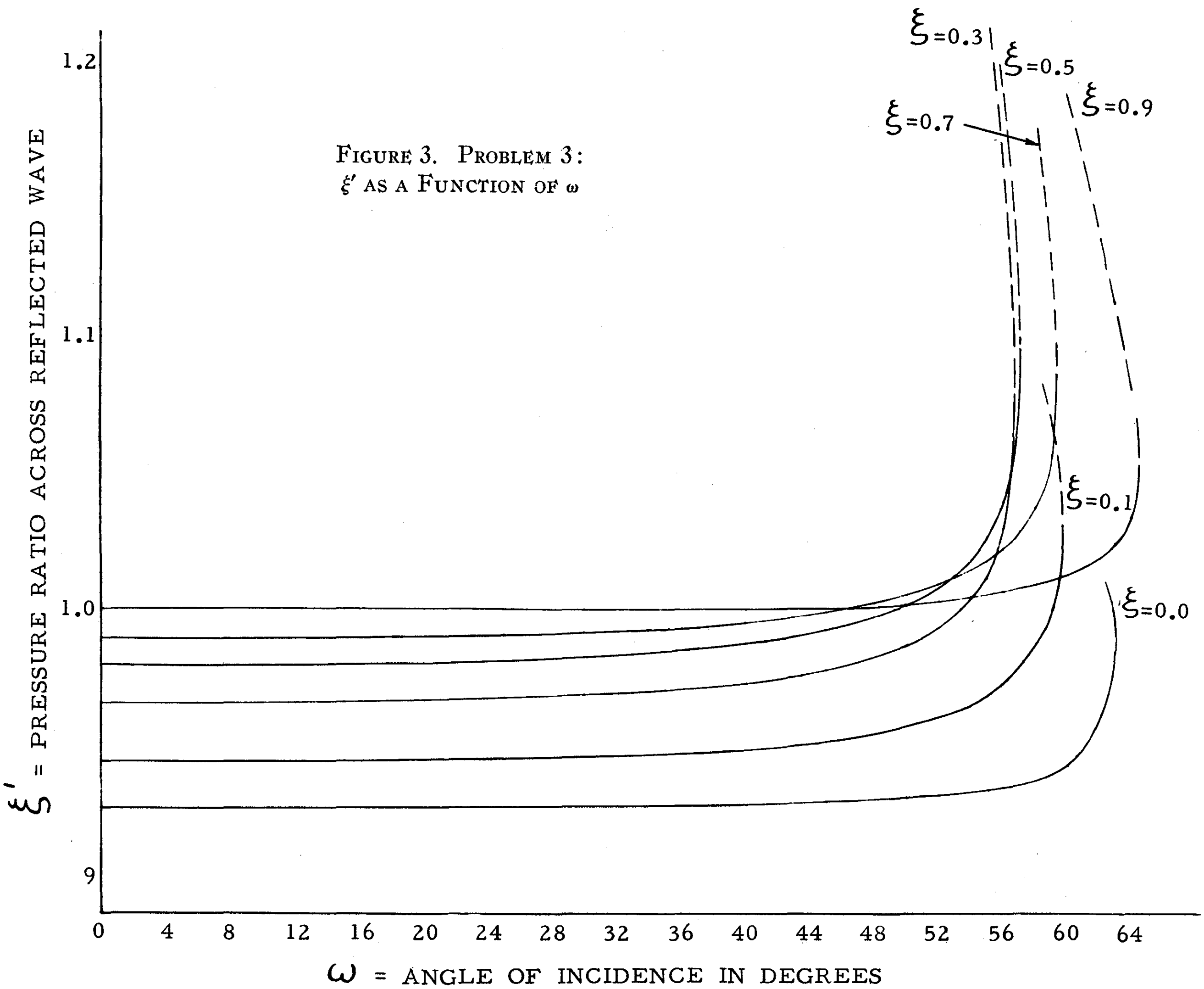

FIGURE 3. PROBLEM 3: ξ' AS A FUNCTION OF ω

TABLE II

Problem	ξ	ω_t	ω_1	ω_L	Problem	ξ	ω_t	ω_1	ω_L
3	1.0	43.089°	90.000°	69.295°	15	1.0	46.911°	90.000°	unreal
	0.9	44.090	74.146	69.295		0.9	48.059	74.146	unreal
	0.7	46.442	65.457	69.295		0.7	50.779	65.457	unreal
	0.5	49.429	61.945	69.295		0.5	54.295	61.945	unreal
	0.3	53.392	61.439	69.295		0.3	59.110	61.439	unreal
	0.1	59.009	64.272	69.295		0.1	66.410	64.272	unreal
	0.0	62.833	67.792	69.295		0.0	72.009	67.792	unreal

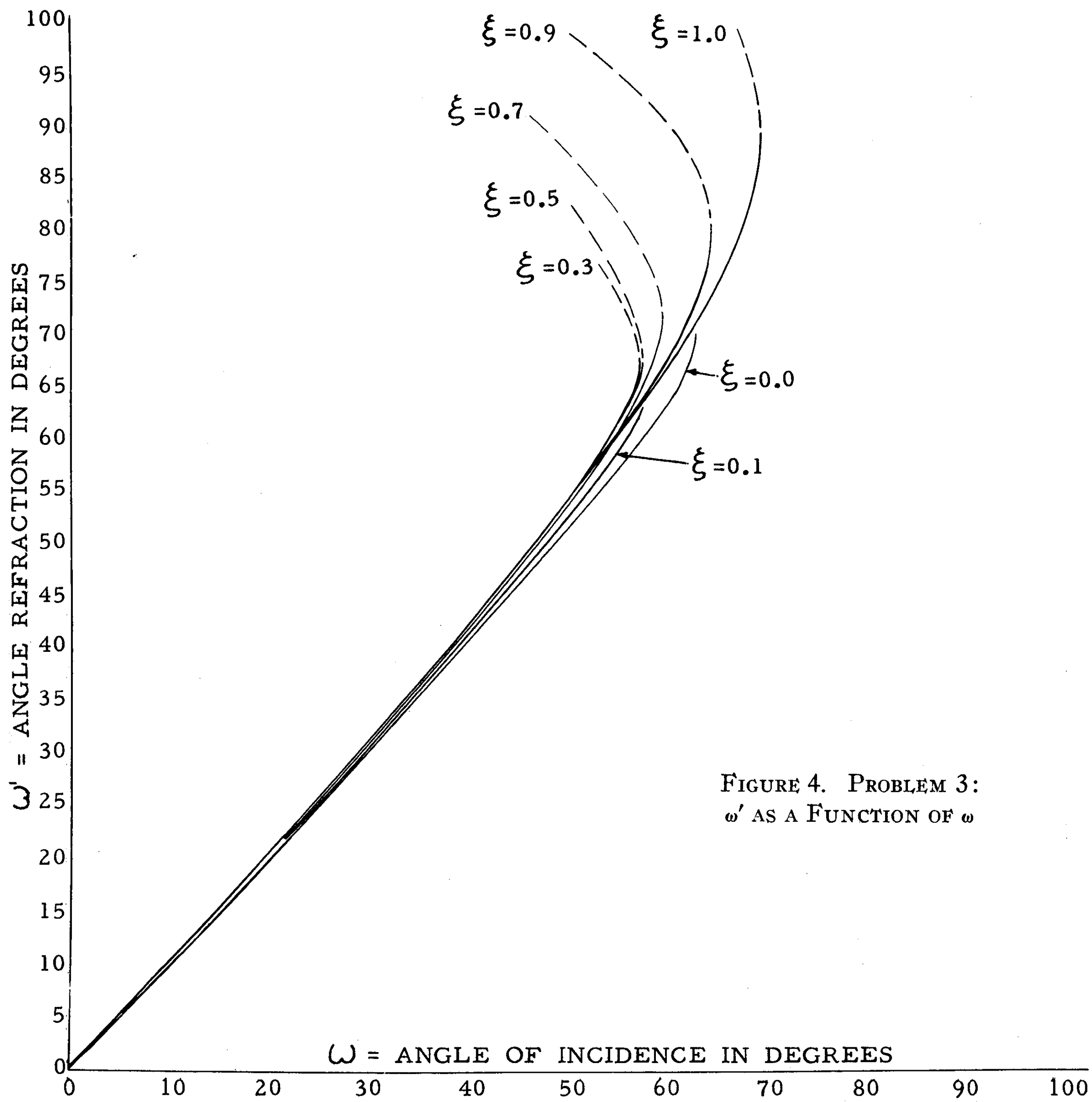

FIGURE 4. PROBLEM 3: ω' AS A FUNCTION OF ω

For Problem 3 a reflected rarefaction wave solution exists at all values of ξ at normal incidence ($\omega = 0$); whereas with the gases interchanged (Problem 15) there is always a reflected shock-wave for normal incidence.

The dependence of the strength ξ' of the reflected wave on the angle of incidence ω for incident shock strengths ξ = 1.0, 0.9, 0.7, 0.5, 0.3, 0.1, 0.0 is plotted in Figure 3. A pressure ratio greater than 1.0 indicates a reflected shock-wave; while a value less than 1.0 indicates a reflected rarefaction wave. We see, as we said before, that for all strengths of the incident shock wave, the type of configuration is of the rarefaction variety for normal incidence ($\omega = 0$).

A typical curve begins at a value of ξ' less than 1, and continues upward until it reaches ω_t, the transition angle, at which point the configuration changes from a rarefaction to a shock-wave variety. It extends upward until it reaches an extreme angle ω_L beyond which no further solutions are possible. The existence of an extreme angle was first demonstrated by J. Von Neumann for the problem of regular reflection from a rigid wall. The similarity of the two problems is here illustrated.

Figure 4 is a plot of ω' (the angle of refraction) against ω (the angle of incidence) for the same problem. For weak incident shock-waves Snell's law of refraction holds. This is the relation used for plotting the curve for ξ = 1. It will be noticed that for any other shock-wave strengths (including infinite strength) the law of refraction does not differ appreciably from Snell's law. This is an interesting result.

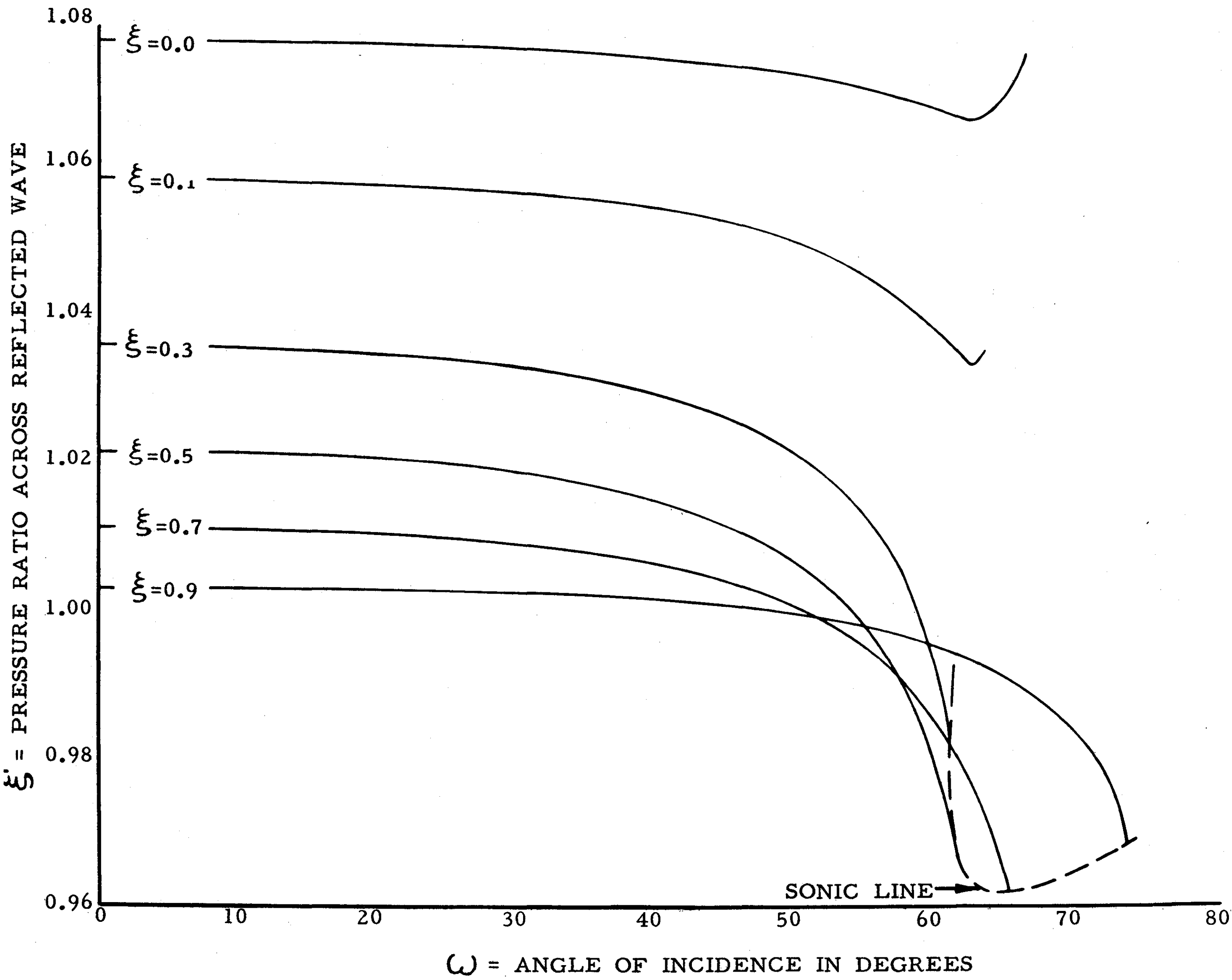

FIGURE 5. PROBLEM 15: ξ' AS A FUNCTION OF ω

For this problem all configurations are of the reflected shock-wave variety for normal incidence, which usually change over to the rarefaction type at larger values of ω. For $\xi = 0.0$ and 0.1, however, no transition takes place, as has been pointed out previously. In that case, the curves reach an extreme angle ω_L, beyond which no further solutions are possible.

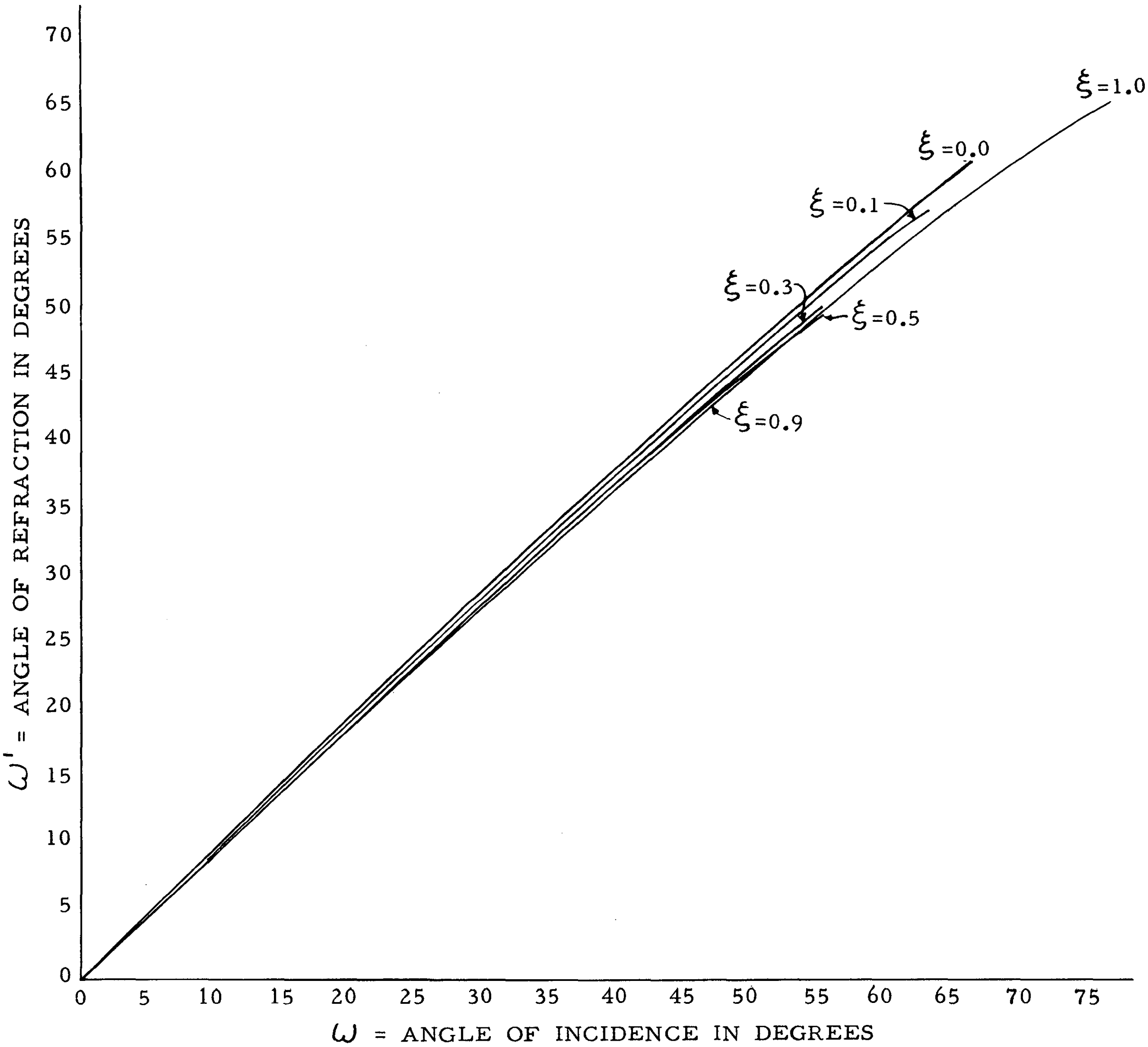

FIGURE 6. PROBLEM 15: ω' AS A FUNCTION OF ω

We will next consider two other problems (10 and 22) which exhibit slightly different properties. In both Problem 10 and 22 there are no real solutions for the transition angle for values of $\xi > 0.66$. This means that for these values of ξ there cannot possibly be any transition from one type pattern to the other. In other words, the refraction pattern which begins at normal incidence must continue throughout the problem.

In the case of Problem 10, the starting pattern is always of the rarefaction type for values of $\xi > 0.66$, but of the shock wave variety for the case $\xi < 0.66$, or where the solution for ω_t is unreal. For Problem 22 exactly the reverse occurs.

Figure 7 is a plot of ξ' (the ratio of pressures across the reflected wave) versus ω. In the case of strong shock-waves we start with a rarefaction type of configuration at $\omega = 0$ and go over to the shock wave variety at ω_t, while for weak incident shock-waves we start with the shock-wave variety, which persists throughout the entire range of ω.

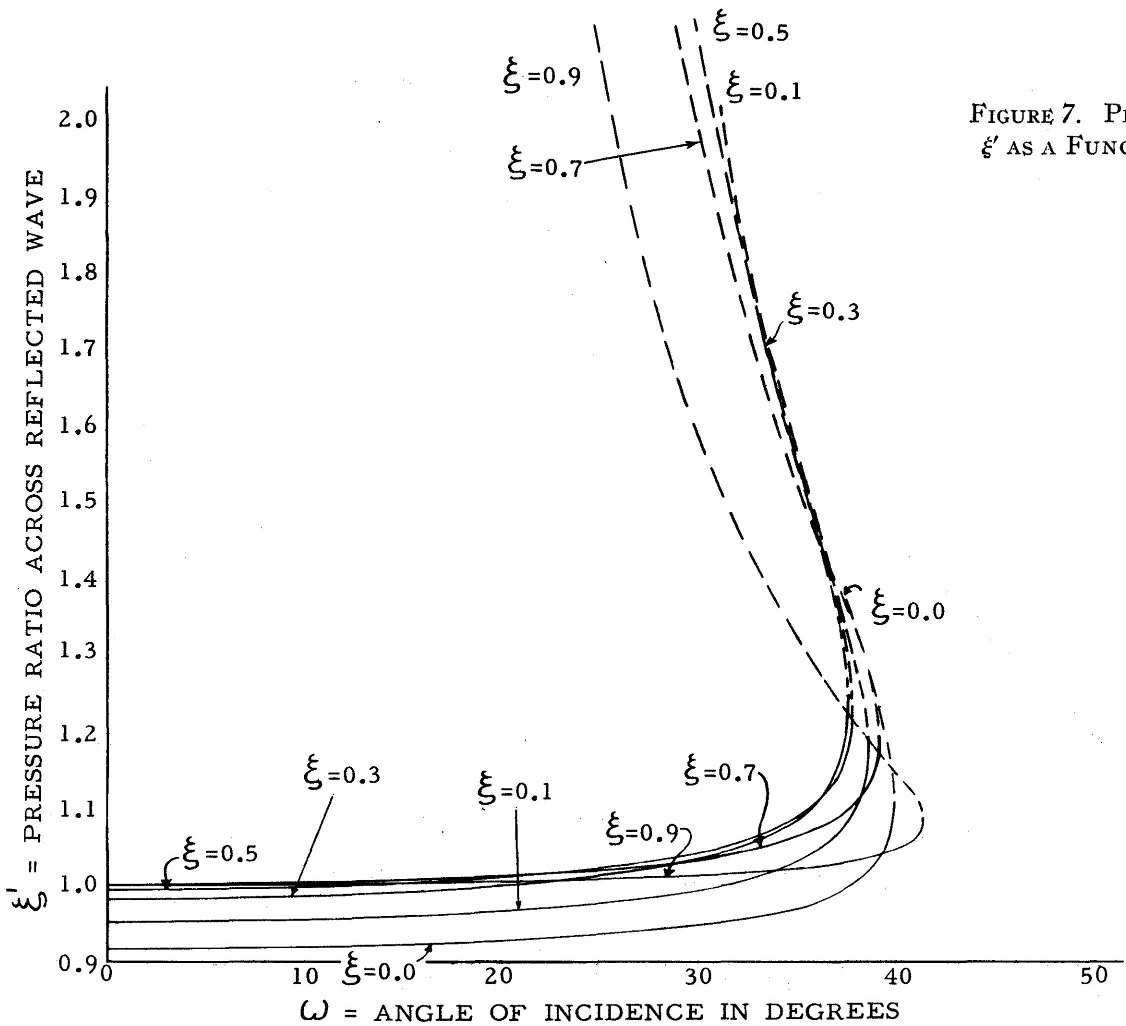

FIGURE 7. PROBLEM 10: ξ' AS A FUNCTION OF ω

TABLE III

Problem	ξ	ω_t	ω_1	ω_L	Problem	ξ	ω_t	ω_1	ω_L
10	1.0	unreal	90°.000	42°.706	22	1.0	unreal	90°.000	unreal
	0.9	unreal	73.554	43.123		0.9	unreal	74.526	unreal
	0.7	unreal	65.378	44.009		0.7	unreal	65.566	unreal
	0.5	11°.820	62.929	44.972		0.5	16°.848	61.439	unreal
	0.3	20.739	64.065	46.025		0.3	29.476	60.005	unreal
	0.1	31.192	70.061	47.183		0.1	44.916	61.294	unreal
	0.0	37.558	77.690	47.807		0.0	55.360	63.431	unreal

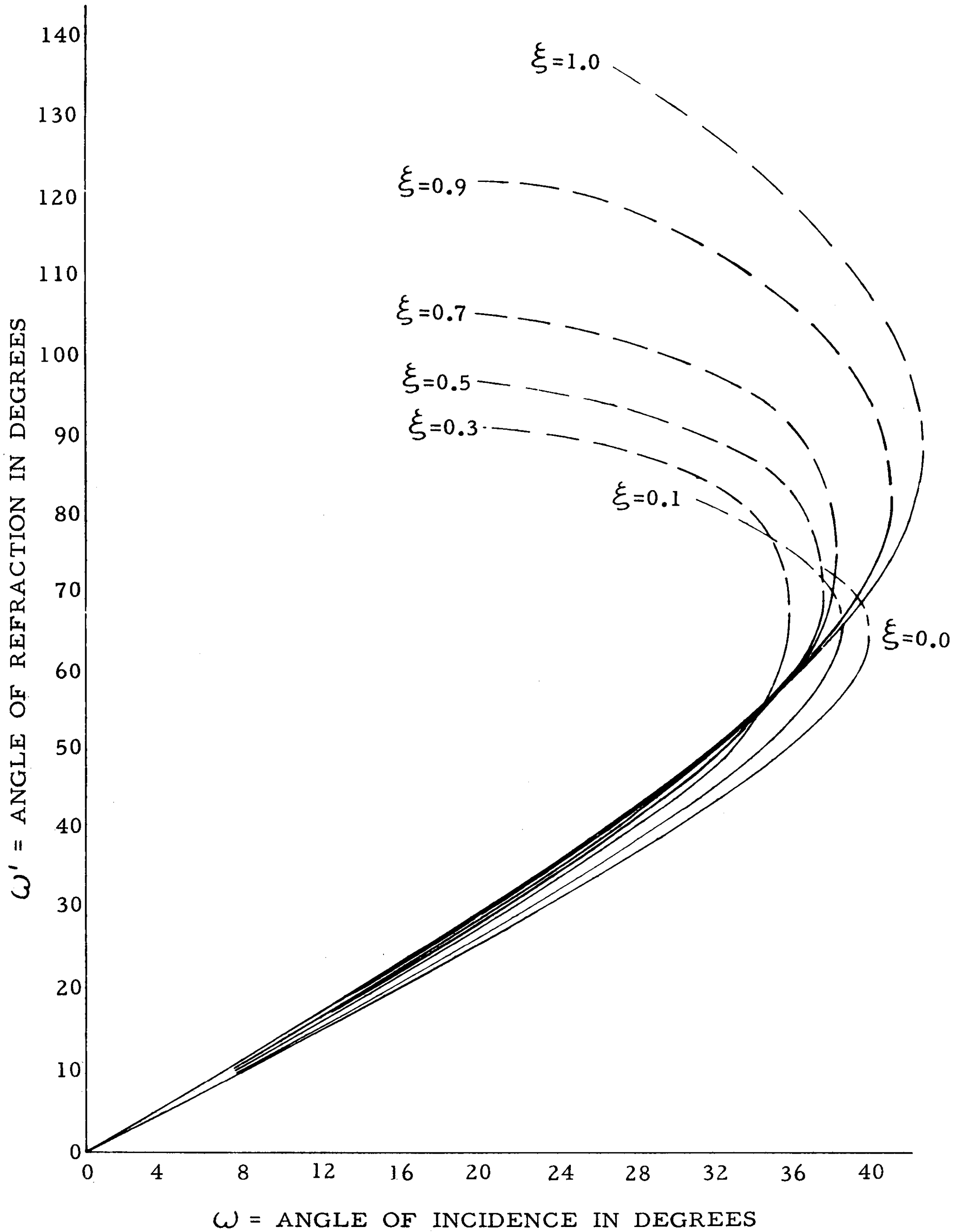

FIGURE 8. PROBLEM 10: ω′ AS A FUNCTION OF ω

A refracted angle of 90° indicates a transmitted wave which is normal to the interface. In acoustics this angle is known as the angle of total reflection, and is the limiting point beyond which no refraction pattern can take place. For stronger shock-waves a refraction angle of 90° does occur mathematically; however, it is on a portion of the curve which we believe to be physically unreal. We must thus expect other types of limiting angles for shock waves of finite amplitude.

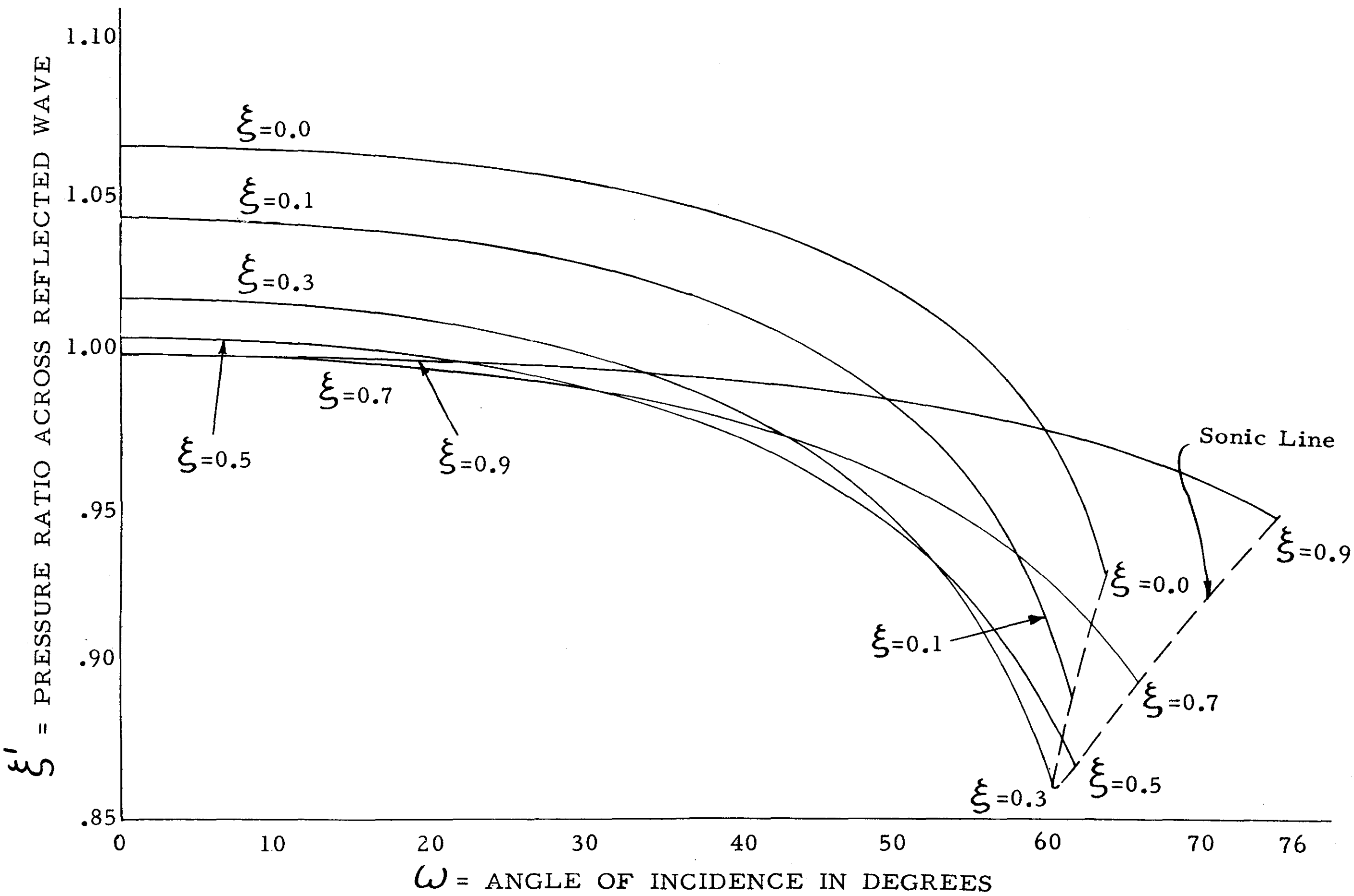

FIGURE 9. PROBLEM 22: ξ' AS A FUNCTION OF ω

In this problem the same gas combination is used as that in Problem 10; however, the gases are interchanged. For strong shock-waves, a shock-wave is reflected at small values of ω and a rarefaction wave is reflected for large values of ω. For weak incident waves, on the other hand, only rarefaction waves are reflected. The range of values of ω for which solutions exist is limited by a sonic line, beyond which the material velocity behind the incident shock wave becomes equal to the velocity of sound.

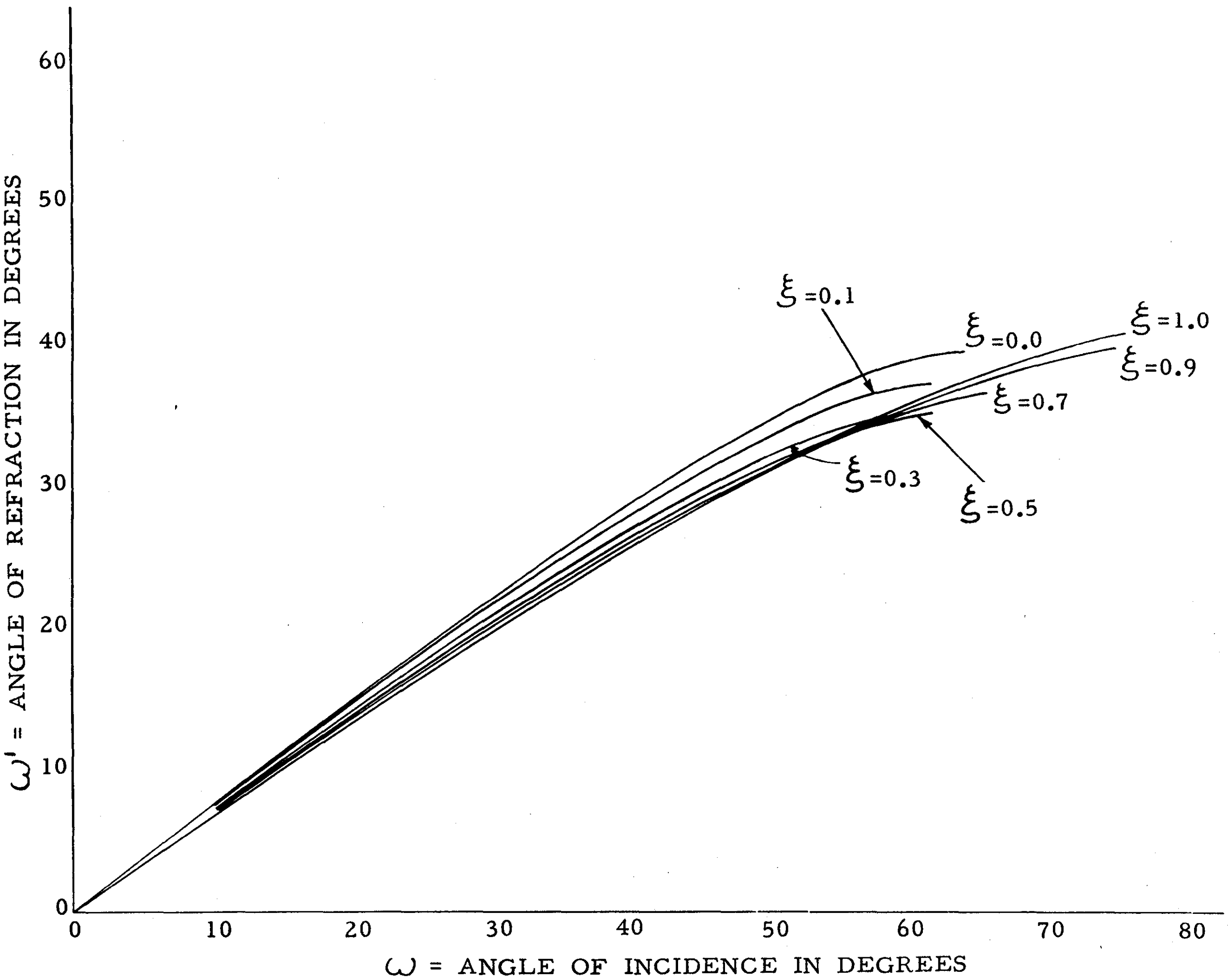

FIGURE 10. PROBLEM 22: ω' AS A FUNCTION OF ω

In Problems 11 and 23 no real solutions for the transition angles ω_t exist at all; hence, the type of pattern which occurs at normal incidence persists throughout the entire range of permissible values of angle of incidence. Thus a shock-wave solution always occurs in Problem 11, while a rarefaction wave solution takes place when the gases are interchanged (Problem 23).

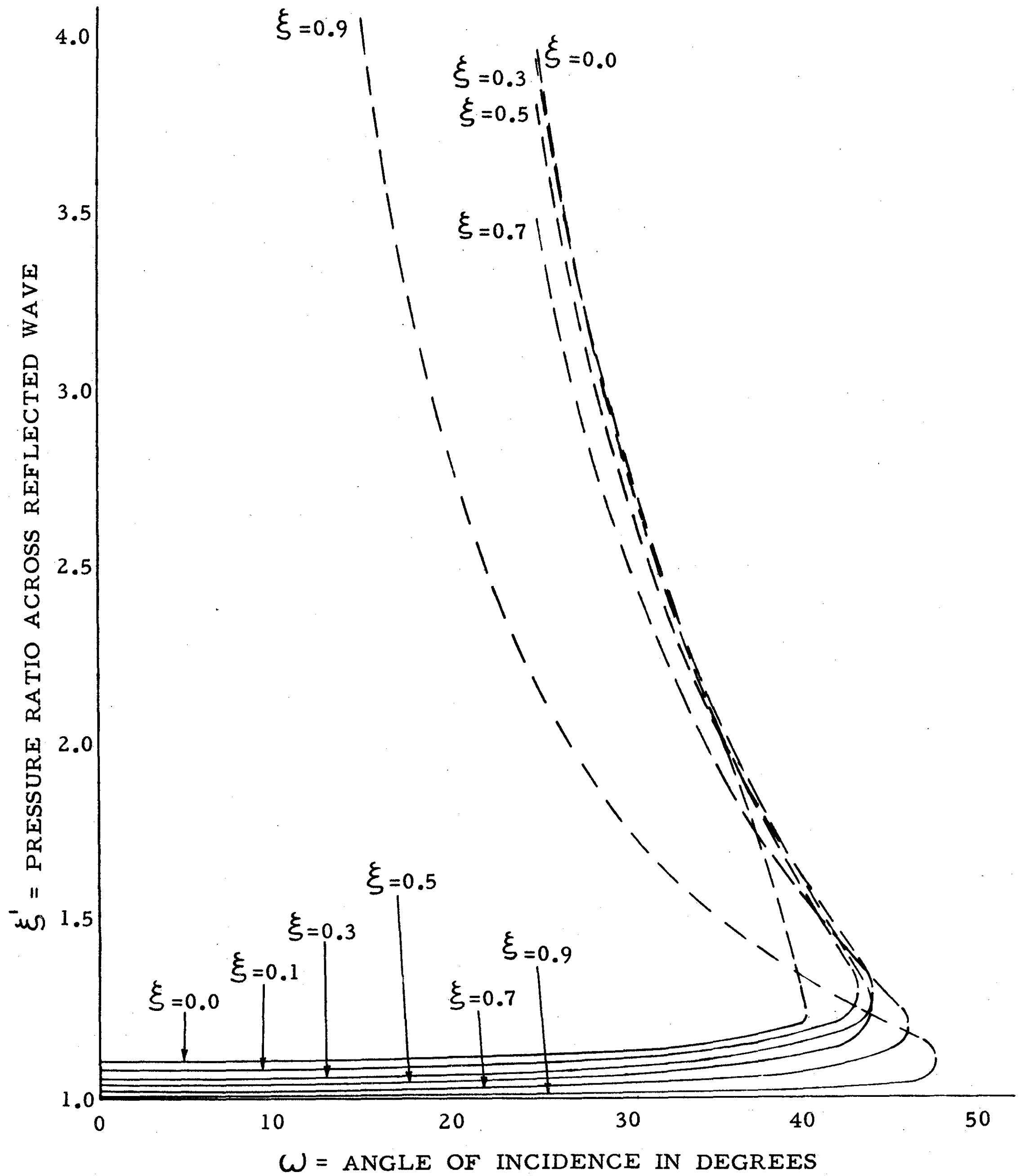

FIGURE 11. PROBLEM 11: ξ' AS A FUNCTION OF ω

The shock-wave solutions do not exist for angles of incidence greater than an extreme angle ω_L.

TABLE IV

Problem	ξ	ω_t	ω_1	ω_L	Problem	ξ	ω_t	ω_1	ω_L
11	1.0	unreal	90.000°	50.769°	23	1.0	unreal	90.000°	unreal
	0.9	unreal	73.554	51.323		0.9	unreal	74.526	unreal
	0.7	unreal	65.378	52.512		0.7	unreal	65.566	unreal
	0.5	unreal	62.929	53.821		0.5	unreal	61.439	unreal
	0.3	unreal	64.065	55.275		0.3	unreal	60.005	unreal
	0.1	unreal	70.061	56.903		0.1	unreal	61.294	unreal
	0.0	unreal	77.690	57.795		0.0	unreal	63.431	unreal

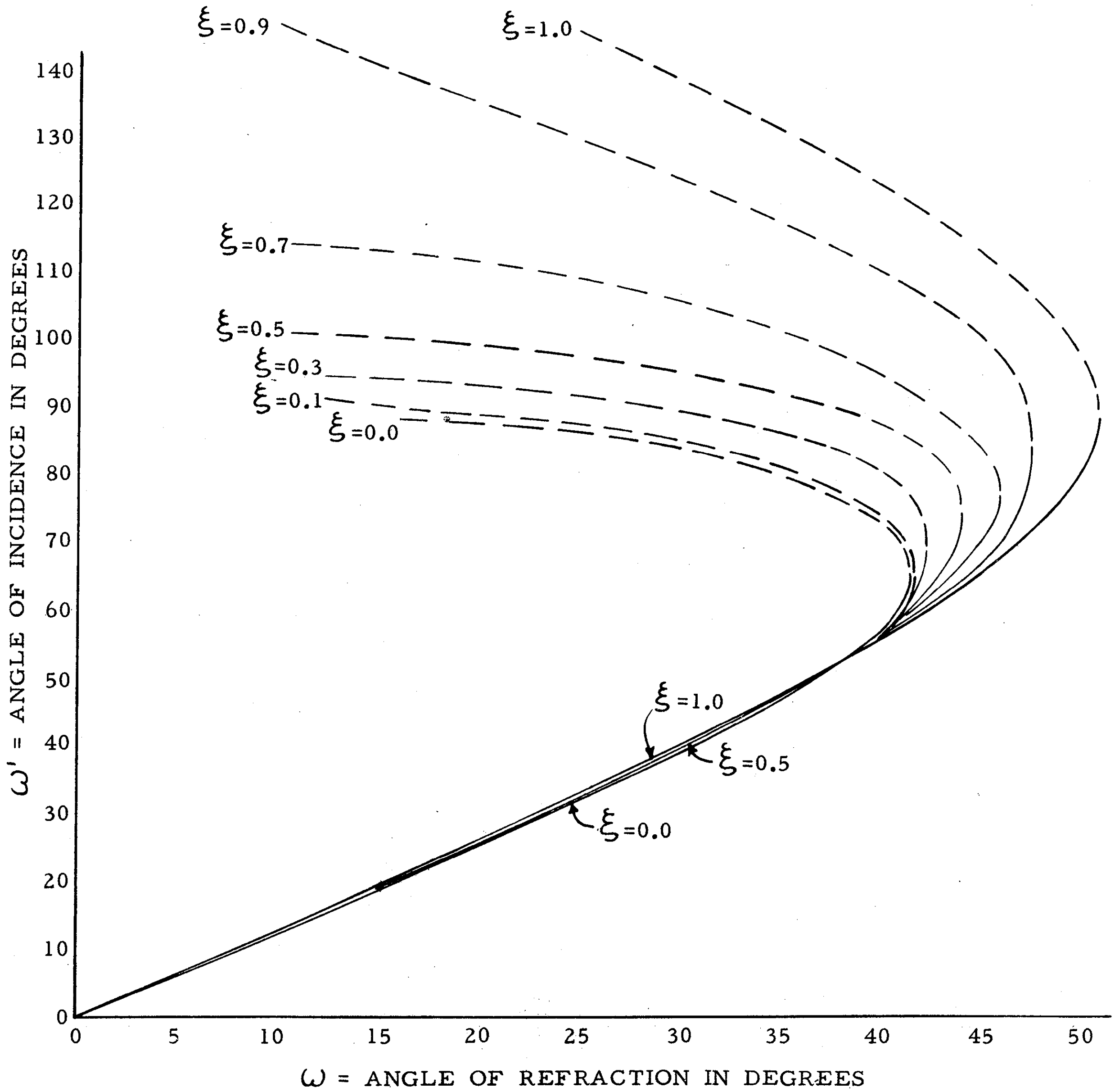

FIGURE 12. PROBLEM 11: ω′ AS A FUNCTION OF ω

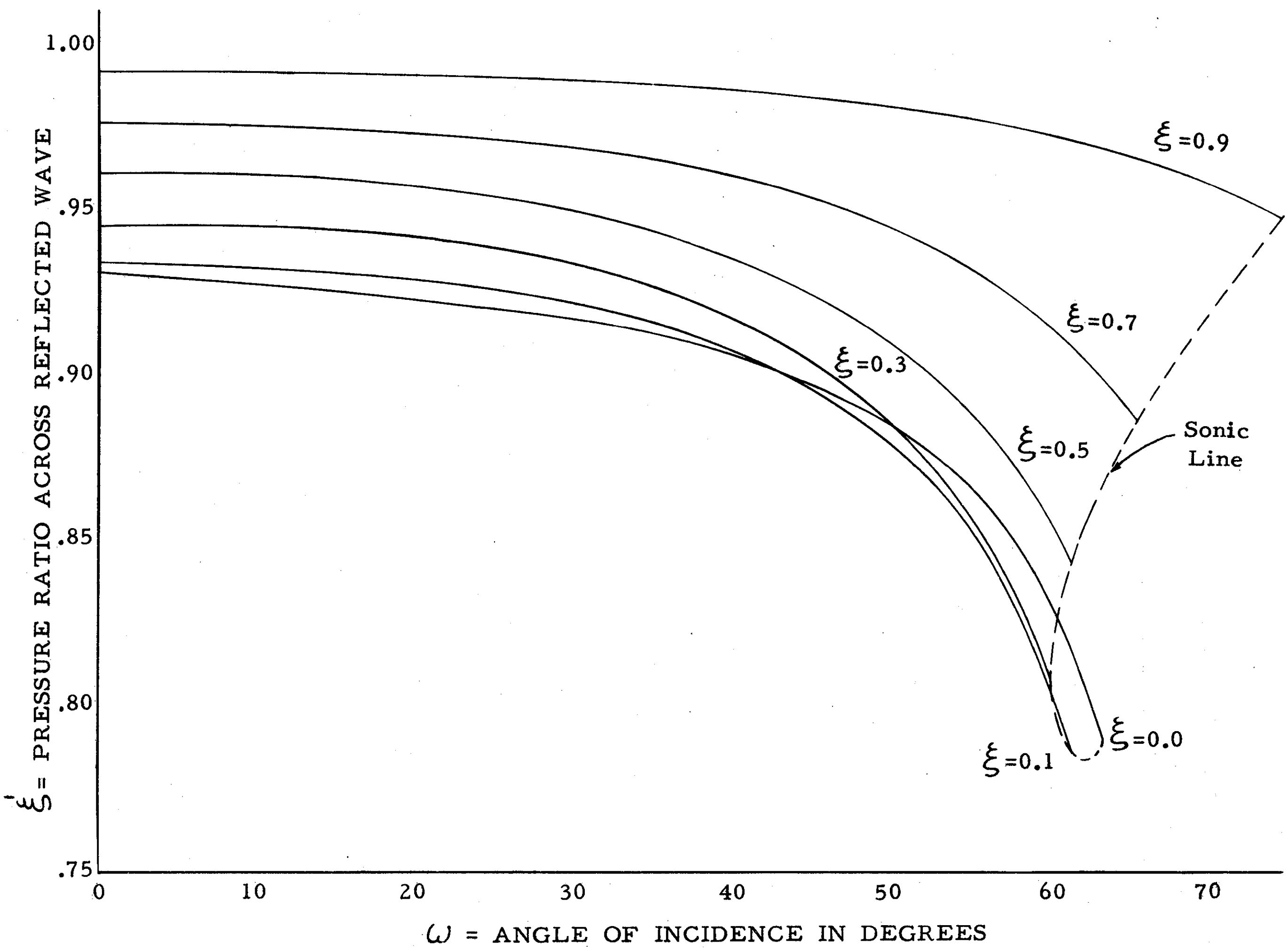

Figure 13. Problem: ξ' as a Function of ω

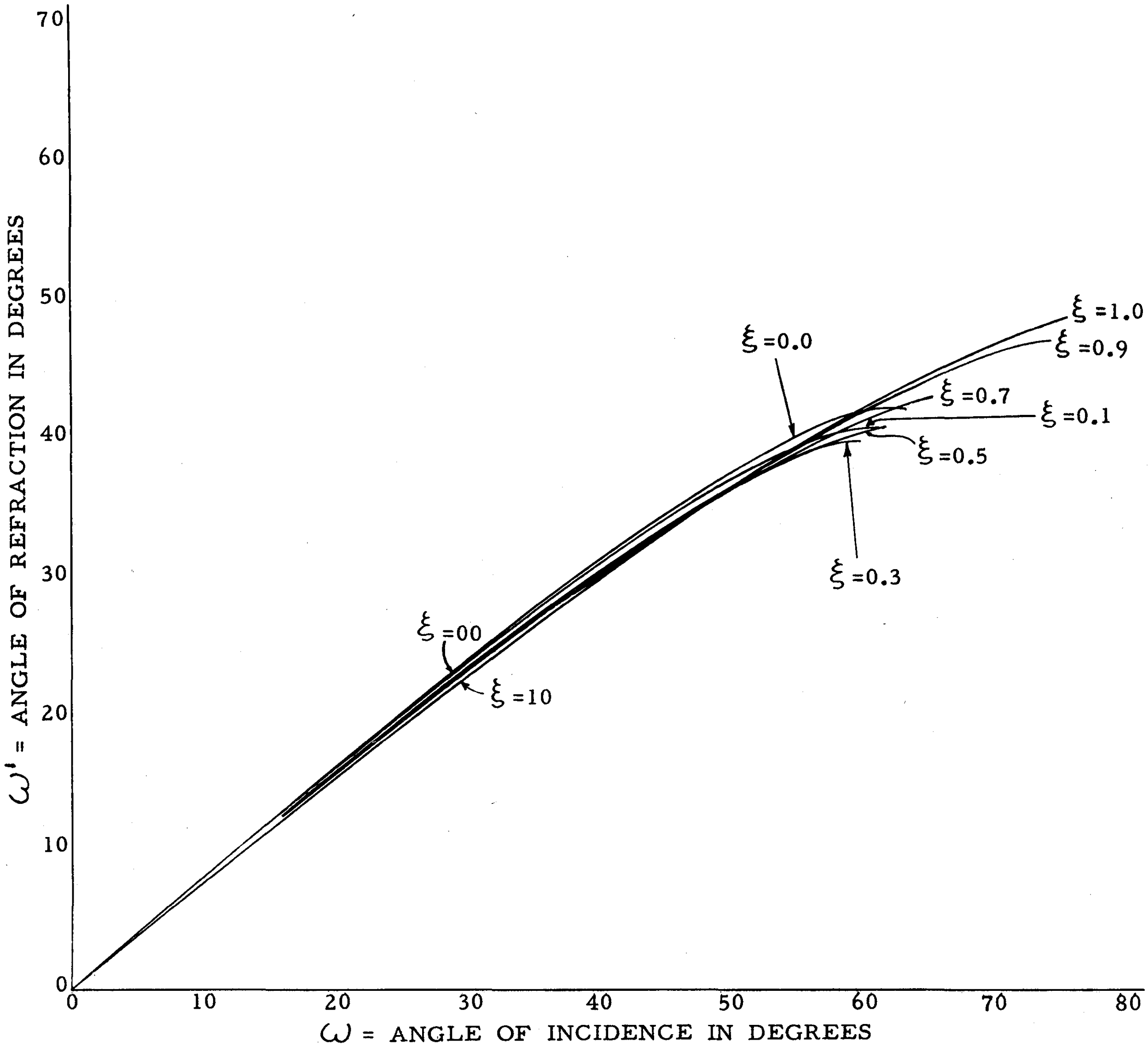

FIGURE 14. PROBLEM 23: ω' AS A FUNCTION OF ω

To sum up, the refraction problem was formulated in terms of a complex system of algebraic equations. In order to solve these equations we first had to carry out a number of preliminary investigations. First, we had five independent parameters to deal with. To take into account all possible variations would probably not be feasible even on a large-scale machine, and would also fill up several large volumes which no one would want to read anyway. We thus selected typical cases of gas combinations; but in each case we obtained solutions for all strengths of the incident shock wave and for all possible angles of incidence.

Secondly, we were faced with the problem of finding out *a priori* which ones of the many possible mathematical solutions were physically plausible. We have solved this difficulty by connecting our solutions with the known solutions for the acoustic case, and for the case of normal incidence.

We still had several other difficulties. We had to consider the two types of configurations, and we had to predict in each case which type of configuration will occur. We also wanted to have a good representation of the various types of characteristic curves that may occur. We wanted to obtain a sufficient number of solutions which we can later compare with experimental results. In all, we solved a total of approximately 3000 points. If a single point were to be solved by hand, it would probably take an experienced computer at least one day. If the entire job were done by hand, it would take one computer approximately fifteen years or a staff of ten computers about a year and a half. In both cases, I believe, the problem would get out of the range of feasibility. It is doubtful if the problem would ever have been solved without the use of a large-scale machine. The actual running time on the machine was approximately sixty hours. Preliminary preparations and coding of the problem for the machine took several weeks. Thus, within a month we actually were able to obtain a solution to a problem which otherwise probably would never have been solved.

I would like to make a few general remarks about the place of machines in the scientific computation field. I do not believe that machines will ever replace the necessity for analysis, or for scientific investigation; however, the automatic calculating machine certainly represents a new and powerful tool which the scientist will be able to utilize in the future in the solution of many difficult and otherwise unfeasible problems. There are really only two large-scale electronic machines in this country at present in actual operation. The ENIAC, at Aberdeen, was the first machine to prove that large-scale computations of complicated mathematical problems is feasible. The SSEC is the second machine. When we used it, it had been in operation for approximately six months, but it had already demonstrated not only that solutions of complex mathematical problems on large-scale electronic machines are feasible, but that these are actually practicable.

In conclusion, I would like to thank the staff of the IBM Corporation for the excellent spirit of cooperation they have shown throughout all phases of this problem, and for the valuable assistance they have offered both in coding the problem and in preparing the problem for the machine. I would also like to mention that this work has been carried out in collaboration with my colleague, Dr. R. J. Seeger, and that the work was sponsored by the Mathematics Branch of the Office of Naval Research.

Computation of Statistical Fields for Atoms and Ions

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I PROPOSE not to go into the theory of the problem that I am going to talk about, but to write down the equations with as little explanation as possible, and show how they were transformed to what seemed to me to be a suitable form for putting on the machines. Then I will show you what the people who coded the problem and put it on the machines gave back to me, and how I had to treat that. I shall be trying to explain the point of view of the person who brings a problem to the machine, to try to show you how much that person has to do and how little he has to do it with.

It happens that in this case the problem is very small compared to the capacity of the machine. I do not need anything like the number of decimal places that the machine has. I do not need anything like the storage that the machine has. The problem could perhaps have been done on a smaller sequence computer. It might have been done, with considerable inconvenience and taking quite a long time, on the relay calculators. However, it is much easier to do it on this machine, and much easier—that is, for me—simply because I do not have to worry about capacity at all, throughout the problem. I just put the equations there and say, "do it," and it turns the solutions out. There is everywhere ample capacity.

$$\nabla^2 (P - 2/\pi)^2/2 = 4P^3/3\pi$$

$$\frac{1}{2}\left(P - 2/\pi\right)^2 \backsim Z/r \qquad r \to 0$$

$$P = 5/2\pi \quad \text{at } r = r_0$$

$$-r^2\, d/dr \left\{\frac{1}{2}(P - 2/\pi)^2\right\} = Z - n \quad \text{at } r = r_0$$

The equation that I want to solve is the above. P is the density of electrons. The problem is supposed to be symmetrical about a single center, the center of an atom; and this is the statistical equation for an approximate field, including the correction term for exchange. Z is the charge of the nucleus, and r is the distance of the electron from the nucleus. The boundary conditions are that $\frac{1}{2}(P-2/\pi)^2$ shall behave like Z/r as r tends to zero; and that at a certain finite distance r_0, P shall take the boundary value $5/2\pi$ while its derivative satisfies the second condition involvinz $Z - n$, where n is the degree of ionization, so that $Z - n$ measures the part of the nuclear change unneutralized by the electrons around it.

We then have a problem in which the outer boundary is not fixed beforehand. The equation is a non-linear ordinary equation, since this is a purely one-dimensional problem. We have one boundary condition at the center; at $r = r_0$ we have the outside movable boundary. It is known from much work on this and on more complicated problems that it is convenient to adopt a logarithmic scale of distance, using a finer mesh where r tends to zero. Therefore, we make a number of substitutions directed towards a logarithmic independent variable in place of r, while leaving a differential equation of the second order without a first order term in it.

The following substitutions were made:

$$\frac{1}{2}\left(P - 2/\pi\right)^2 = \psi = 9\pi^2\phi/128 = e^{x/2}\chi$$

$$r = e^{-x}, \quad (x - x_0) = y, \quad \chi = e^{7x_0/2}\omega\,.$$

Going to a logarithmic scale, if $r = e^{-x}$, $\psi = e^{x/2}\chi$ removes the first order term from the resulting equation. It is convenient also to scale the problem so that the movable boundary is fixed, writing $x - x_0 = y$, at $y = 0$. Writing χ as shown makes the coefficients of the final equation simpler.

This gives us the equations in the following form:

$$\frac{d^2\omega}{dy^2} = \frac{1}{4}\omega + e^{-5y/2}\left\{e^{y/4}\omega^{1/2} + \gamma\right\}^3$$

where

$$\gamma = 16\, e^{-2x_0}/3\pi^2\,,$$

with

$$e^{-y/2}\omega \to 128\, e^{-3x_0} Z/9\pi^2 \qquad \text{as } y \to \infty$$

$$\left.\begin{aligned} &e^{y/4}\omega^{1/2} = 4\, e^{-2x_0}/3\pi^2 \\ &e^{-y/2}\left(\frac{d\omega}{dy} + \frac{1}{2}\omega\right) = \frac{128}{9\pi^2} e^{-3x_0}(Z - n) \end{aligned}\right\} \text{for } y = 0.$$

Here $y \to \infty$ corresponds to the radius going to zero.

What we want are solutions for various values of Z and n. If we start integrating the differential equation from the outside, $y = 0$, n being known, the value of one other parameter γ is all that is needed to start the solution, and we will have for each value of n a one-dimensional family of solutions. When we have integrated in sufficiently far to see what happens as $y \rightarrow \infty$, we can compute to what value of Z this value of γ was appropriate. The total number of solutions required is reduced by this scaling process to a manageable number.

We wish to compute not only the variation of the potential and the distribution of electrons as you go onwards in the atom, but also over-all integrals of these expressions which can be used to give various physical properties of the solution. It turns out that we would like six integrals over the solution, which, however, have two relations between them—identical relations if the boundary conditions are satisfied.

One of these is a double integral. It represents the mutual potential energy of all the electrons. It is not convenient to calculate the double integral, but since we have two identical relations, if we calculate the other four integrals, we can calculate this one from them, and still have an identical relation between them, which will act as a very good check on the accuracy of the results.

I will put down the forms of two of these integrals:

Total Charge $E = \frac{4}{3\pi}\int P^3 r^2 dr$,

Kinetic Energy $K = \frac{2}{5\pi}\int P^5 r^2 dr$.

Now P is not given us directly by the solution, but is proportional to $e^{y/4}\,\omega^{1/2} + \gamma$, which will occur in the integrals to a whole number power, 3, 4 or 5. It is convenient to divide the factors beyond the third power into their terms, giving four independent integrals to obtain numerically.

I then want to get, with a reasonably small number of steps, a reasonably accurate solution. I will be quite satisfied with four digits if I know they are accurate, because after all the statistical field is only a rough approximation to what actually goes on in an atom. However, in this machine you do not save anything by using less than a nine-digit register, and so things were set up in nine-digit registers. It turns out that without taking any special care we get five-digit accuracy in the answers. In order to have a reasonably small number of steps I decided that we must use a more accurate type of integration in which the error term is in the sixth order of differences. That somewhat complicates the equations. It does not complicate very much the work on the machine, because it is necessary with this differential equation to carry out an iteration for each step, to find the new value of ω, in any case.

If you replace $d^2\omega/dy^2$ by an expression in terms of differences, you can solve the equation. If you replace it merely by the second difference, you can go straight ahead substituting the new value of ω in the right-hand side. All that you would have to do would be the square root (a subsequence on this machine). However, if I make use of the more accurate formulas, I do not have values of ω in my table which I am building up, but values of $\omega - w^2/12 \cdot d^2\omega/dy^2$, w being the interval. $d^2\omega/dy^2$ is related to ω by the differential equation, and ω must be inferred. This is done by regarding the table value as an approximation to ω, substituting this in the differential equation, evaluating $d^2\omega/dy^2$, and iterating until an accuracy set in a certain decimal place is reached. In each iteration it is necessary to take a square root.

Here we have a main sequence; then a sub-sequence to get a new value of ω by iteration, in each step of which a sub-iteration is done to find a square root. When a tolerance is reached, we go back to the main sequence, accumulating sums which approximately represent the integrals desired.

It turned out when we first set this up that we did not get sufficient accuracy near the beginning of the solution because the interval there was taken too large. So the problem was recoded, starting with one-sixteenth this interval, going the first eight steps with the smaller interval, and then doubling the interval; then four steps and doubling the interval again, and so on, until we came back to the original interval. This gives enough accuracy. It was no more difficult to set all these interval doublings than to set up one or two doublings, since the process is the same. Of course, it takes a little longer running the problem on the machine, but only a small fraction of the total time, since the total number of steps is of the order of seventy.

Well, when this has been done the machine provides me with sheets from its tabulation stations. It has two tabular outputs, as shown in Figure 1. One is from the first printer; this contains the main part of the integration. It contains the running sum, $\omega - w^2/12 \cdot d^2\omega/dy^2$, and the result of taking the square root, putting it in $(e^{y/4}\,\omega^{1/2} + \gamma)$, and cubing it. Those expressions together are enough to make it easy to check any line by multiplying a few numbers together.

The other printer gives me progressive totals that give the sums which I want to compute. Actually, you do not need these progressive totals if a uniform interval has been used throughout. However, at the places where the intervals were changed these sums are simply progressive sums. A correction had to be made to the value of the integral.

I now wish that I had asked for that correction to be put in by the machine. It could have been sequenced and would have saved me a good deal of trouble in processing

$\gamma = 2.9000$

$\eta_0 = 0.525612$

$\eta_1 = 0.525217$

READ UP

χ_i	η_i	$\{e^{\gamma/4}\omega^{1/2}+\gamma\}^3$	$(e^{-\gamma/4}\chi^{1/2})$
92484 627526	92382 472634	890 2221	9 6169
82426 839847	82335 794389	890 3322	9 6169
73462 805777	73381 661635	890 4433	9 6169
65473 578095	65401 258545	890 5269	9 6168
58353 144691	58288 690091	890 6655	9 6168
52007 021802	51949 576877	890 8327	9 6168
46351 000303	46299 802795	890 9189	9 6168
41310 028310	41264 398852	891 1661	9 6167
36817 215256	36776 548381	891 3084	9 6167
32812 944345	32776 700420	891 6107	9 6166
29244 081431	29211 779523	891 8606	9 6165
26063 269994	26034 481477	892 1390	9 6164
23228 302743	23202 645610	892 4441	9 6163
20701 561558	20678 695350	892 8339	9 6162
18449 518341	18429 139624	893 1152	9 6160
16442 290088	16424 128455	893 6127	9 6157
14653 242436	14637 056886	894 0064	9 6154
13058 636224	13044 211966	894 5078	9 6150
11637 312475	11624 458108	895 1445	9 6145
10370 411738	10358 956671	895 7576	9 6139
9241 123783	9230 916008	896 4270	9 6131
8234 464653	8225 368686	897 1243	9 6121
7337 077825	7328 972950	897 8781	9 6108
6537 057103	6529 835736	898 7160	9 6093
5823 788675	5817 354950	899 5543	9 6073
5187 810396	5182 078888	900 4493	9 6049
4620 686358	4615 580941	901 3447	9 6018
4114 895114	4110 347964	902 2404	9 5980
3663 730114	3659 680781	903 1098	9 5932
3261 210933	3257 605558	903 8145	9 5871
946084	945359	48 1095	9383
864423	863698	47 3673	9001
791432	791249	46 7376	8643
758182	757999	46 4698	8476
727102	726919	46 2296	8315
698196	698013	46 0289	8163
671469	671286	45 8596	8019
646929	646746	45 7292	7885
624585	624402	45 6408	7762
604448	604265	45 5151	7649
586549	586499	45 5189	7549
578428	578378	45 6178	7503
570868	570818	45 6525	7461
563872	563822	45 6186	7422
557442	557392	45 7560	7386
551581	551531	45 8252	7353
546292	546242	45 9134	7325
541578	541528	46 0096	7299
537424	537416	46 1294	7278
535574	535566	46 1108	7269
533870	533862	46 2607	7261
532312	532304	46 3302	7253
530901	530894	46 4078	7247
529637	529629	46 4050	7242
528521	528513	46 5665	7237
527553	527545	46 6522	7233
526743	526726	46 7456	7231
526074	526057	46 8429	7230
525555	525538	46 9442	7230
525187	525170	47 0459	7230
524970	524953	47 1592	7233
524905	524888	47 2691	7235
524993	524976	47 3069	7239
525234	525217	47 5086	7244

$\sum we^{-\gamma/2}\cdot\{\}^3$	$\sum we^{-3\gamma/2}\cdot\{\}^3$	$\sum we^{-\gamma(\,)}\cdot\{\}^3$	$\sum we^{-\gamma}\chi\cdot\{\}^3$
752 4384	92 5137	1194 3290	45499 8665
752 4384	92 5137	1194 3290	45499 8665
752 2334	92 5137	1194 3290	45480 9048
752 0034	92 5137	1194 3290	45459 6259
751 7453	92 5137	1194 3290	45435 7620
751 4557	92 5137	1194 3290	45408 9571
751 1307	92 5137	1194 3290	45378 8526
750 7659	92 5137	1194 3290	45345 0900
750 3566	92 5137	1194 3290	45307 2225
749 8972	92 5137	1194 3290	45264 7139
749 3816	92 5137	1194 3290	45217 0246
748 8030	92 5137	1194 3290	45163 5280
748 1536	92 5137	1194 3290	45103 5058
747 4247	92 5137	1194 3290	45036 0601
746 6066	92 5137	1194 3290	44960 3808
745 6883	92 5137	1194 3290	44875 4723
744 6575	92 5137	1194 2397	44780 1684
743 5004	92 5137	1194 1503	44673 2030
742 2014	92 5136	1194 0609	44553 1272
740 7431	92 5135	1193 9714	44418 3128
739 1059	92 5134	1193 8819	44266 9439
737 2676	92 5133	1193 7027	44097 0187
735 2035	92 5131	1193 5234	43906 2590
732 8857	92 5128	1193 2543	43692 1154
730 2829	92 5124	1192 8951	43451 6636
727 3598	92 5118	1192 5356	43181 7792
724 0770	92 5110	1191 9959	42878 8093
720 3900	92 5098	1191 3656	42538 7096
716 2490	92 5082	1190 5544	42156 9000
711 5981	92 5059	1189 5619	41728 4260
706 3747	92 5026	1188 2975	41247 7007
5 4358	5 1186	4 0757	3 2397
4 9596	4 6942	3 6878	2 8845
4 6428	4 4098	3 4322	2 6568
4 3265	4 1238	3 1826	2 4395
4 0104	3 8360	2 9386	2 2278
3 6944	3 5461	2 6955	2 0260
3 3781	3 2539	2 4577	1 8294
3 0613	2 9591	2 2204	1 6377
2 7437	2 6614	1 9833	1 4508
2 5046	2 4357	1 8100	1 3140
2 3448	2 2843	1 6914	1 2228
2 1846	2 1320	1 5727	1 1315
2 0240	1 9787	1 4585	1 0447
1 8629	1 8244	1 3441	9578
1 7013	1 6690	1 2295	8707
1 5391	1 5125	1 1147	7835
1 3762	1 3548	9997	6961
1 2535	1 2356	9121	6315
1 1715	1 1558	8521	5899
1 0893	1 0757	7920	5483
1 0069	9952	7318	5066
9243	9144	6715	4648
8415	8332	6111	4230
7585	7517	5506	3811
6753	6698	4900	3391
5918	5875	4292	2970
5081	5048	3683	2548
4241	4217	3073	2126
3398	3382	2461	1703
2553	2543	1848	1279
1705	1700	1234	0854
0854	0852	0618	0428

FIGURE 1

these solutions. It turns out that if you try to make a rough check on results by identical relation between the integrals, the check is not nearly accurate enough unless you have added in these extra terms to the sums. I had hoped that I would not have to do this for the originally assumed values of γ, but it turns out that to get a good check one must.

Well, then, these solutions were made originally for $n = 0$, for a wide set of values of γ. Gamma equals 1, 2, 3, 4, 5, 6—I think that is as far as it went. For each of these we computed the corresponding value of Z. Actually, that was computed as we went along, and we found that this covered the whole of the periodic table. We then took the part which included the whole of the periodic table and took even intervals among these values of $\gamma = 2.1$, 2.2, and so on, up to 5.9. And with the integrations of the differential equations at this distance apart in γ, we found the values of Z. That gave a table in which inverse interpolations gave us new values of γ which would lead to assigned values of Z accurate to five decimals. So that as it turns out, without too much work, it was possible to get for any values of n and Z—and we have done this also for $n = 1$ and 2—by an inverse interpolation, the corresponding value of γ, to put that in as a starting value, and to compute the whole field for that particular case.

To compute the field by integration, once the sequence is set up, is a good deal easier than to get the whole field at each point by interpolation among the solutions that have been made, although that would be perfectly possible. I think actually we shall probably tabulate and publish just certain selected solutions. However, we hope to be able to do enough so that anyone could interpolate to find any solution that they might want.

It should again be realized that the statistical fields only give a rather rough order of approximation to the properties of the atomic ion. You really want to use these as a starting field to start after the real approximations in any particular case in which you are interested. I think it is clear that it would be too much work to work out complete sets of Hartree approximations for every possible atom. If you have once got a statistical field, that is a good start toward obtaining the Hartree field. You could probably get the Hartree field with at most two successive approximations of the Hartree type.