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Topics in Model Theory

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Abstract: The concept of "free" as in free group and free semi-group is extended to arbitrary first order theories. Every consistent theory has free models. Some problems of obtaining a categorical theory of models are discussed.

Part I Definition of Free Models

Model theory is usually based upon a system of first order logic using predicates, variables, and quantifiers. Freyd[1] considers the category of models for such a theory. This requires that homomorphisms be defined between models. It turns out to be overly restrictive to define a homomorphism as a map that preserves all predicate relations (for example if inequality is preserved, then all homomorphisms are monomorphic,) therefore, he defines it as a map that preserves only certain predicates in a specified list. Thus the category is established with respect to a theory and a list of relations within that theory.

We get around this unpleasant feature by using a different sort of calculus, namely the free variable equation calculus with equality. This system is as powerful as the usual predicate calculus and can be used to formulate, for example, the theory of arithmetic as in Goodstein[2].

Definitions: (for the free variable equation calculus)

Terms are built by composition from variables, constants, and functions. Each function has a definite number of arguments which is 1.

If t_1 and t_2 are terms, then $t_1=t_2$ and $t_1\neq t_2$ are literals.

A formula is a Boolean combination of literals.

A theory is a set of formulas. They may be regarded as axioms. We do not require that the theory be finite or countable.

A model for a theory is a non-empty set of elements together with an interpretation for the constants and functions in the theory such that:

a. the constants are identified with members of the set

b. the functions are identified with actual functions of n-tuples of the set into itself

c. the axioms are satisfied

d. the axioms of equality including the replacement schema are satisfied

Let T be any theory, and let S be any set such that the cardinality of S is at least that of the set of constants in T. Then we shall define the Herbrand universe of T with generators in S. The notation for this is H(T,S). We define H_0 to be the set S with certain elements of S identified as the constants named in T. Then H_{n+1} is defined as H_n union $\{f(t_1 \ldots t_k)\}$ where f is a function in T with k arguments, and t_1 through t_k are terms in H_n. H(T,S) is then defined as the union of the H_n.

Let H(T,S) be a particular Herbrand universe for T. Then x is a normal model for T if x is an equivalence relation on the terms in H(T,S) such that the axioms of equality and the theory T are satisfied. We shall refer to this equivalence relation as $=_x$. For any t_1 and t_2, either $t_1 =_x t_2$ or $t_1 \neq_x t_2$.

A homomorphism between models is a map from the underlying set of one to the underlying set of the other that maps constants onto themselves and preserves functional composition ($\phi f(t_1 \ldots t_k) = f(\phi t_1 \ldots \phi t_k)$). Strictly speaking, there are two functions "f" involved here, but both are represented by the same function symbol in T. An isomorphism is a homomorphism that is one to one and onto.

Every model is isomorphic to a normal model. This can be seen by taking the underlying set of any model as a generating set. Then there is a normal model which has H_0 as distinct elements with the composite terms each being equivalent to some generator in H_0. We shall consider only normal models in the rest of this paper and call them models.

The set of all models defined as equivalence relations on H(T,S) will be called M(T,S). If T is consistent, then M(T,S) is non-empty.

If x and y are in M(T,S), then we say x   y if and only if whenever $t_1 \neq_x t_2$, then $t_1 \neq_y t_2$. If x   y and y   x, then y=x. Also, if x   y and y   z, then x   z. Thus   is a partial ordering. Let R be any totally ordered subset of M(T,S). Define x_0 by $t_1 \neq_{x_0} t_2$ if and only if $t_1 \neq_x t_2$ for at least one x in R. That x_0 is a model follows from the compactness theorem, for if x_0 is inconsistent, then this must come from finitely many literals. But each of these literals is in some x in R, and since they are finite in number, and R is totally ordered, they are all in the largest of these. Thus any finite set of literals in x_0 is consistent, and therefore x_0 is consistent. x_0 is an upper bound for the elements of R.

By Zorn's lemma, M(T,S) has maximal elements, and these we call the free models of M(T,S).

Theorem: Every consistent theory has free models.

Part II Examples of Free Models

The notion of free defined in part I coincides with the usual special cases. A free semigroup is the set of strings on its generators. The theory of semigroups is simply $x\cdot(y\cdot z)=(x\cdot y)\cdot z$. The Herbrand universe on any set of generators is all possible compositinns using "·". But the theory forces them into equivalence classes so as to be associative.

Free groups almost coincide with the usual definition. The free group on one generator is the trivial group because the generator gets identified with the identity specified in the theory. The free group on two generators has one generator in the usual sense, and turns out to be the integers. Free abelian groups turn out to be n-tuples of integers (or maps from S into the integers for infinite S).

Free fields require at least two generators. The smallest one is the rationals (Q). The field on three generators is usually known as Q(X) and contains rational functions of one indeterminite.

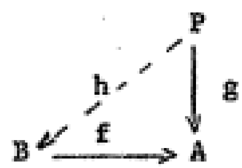
The theory of integers was proven incomplete by Godel, and therefore has non-standard models. Nevertheless, its only free model on one generator is the standard one because every member of the Herbrand universe is probably equal to some member of the sequence 0, 0', 0''...The non-standard models need additional generators.

A complete theory has only isomorphic models, and this one is free. Many theories, such as the theory of groups, are not complete, but have the interesting property that for any S of sufficiently high cardinality, M(T,S) has exactly one free model. These theories might be said to be completely structured. I would like a better name for this.

Part III Categorical Model Theory

The hope of obtaining a categorical model theory seem rather dim at present. The models of a theory do form a category. The notions of completeness and consistency are categorical ones. In fact, all complete theories have isomorphic categories. There is a canonical function from the category of any theory to the category of any subtheory. (The null theory has as its category all non-empty sets and all maps between sets.)

I can find no categorical definition coinciding with the non-categorical definition of free. Projective seemed like a reasonable hope. It may even work for certain theories such as group theory. But it fails in the case of fields where all homomorphisms are monomorphisms. Thus if



where f is epimorphic, then f is an isomorphism and h certainly exists. Thus every field (P) is projective.

This raises the general question as to just how much of a theory can be recovered from its category of models. I have no idea as to the answer.

This investigation is being continued.

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[1] Peter Freyd, Abelian Categories, Harper and Row, 1964. pp.91-93.

[2] R. L. Goodstein, Recursive Number Theory, North-Holland, 1957.