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Topics In Model Theory

Michael Levin

ABSTRACT

The concept of free as in "free group" is generalized to any first order theory. An interesting class of homomorphisms between models is discussed. Relations between model theory and abelian categories are discussed speculatively.

This paper represents an incomplete study and may contain serious errors.

A knowledge of model theory, and of M.I.T. course 18.892 in particular is assumed.

Defn. 1 A *layer* is defined to be a 4-tuple $\langle \Pi, T, S, \delta \rangle$ where $\langle \Pi, T \rangle$ is a similarity type, S is a non-empty set, and δ is a mapping from the 0-ary function letters into S. (If $T(k)=0$, then $\delta(F^k) \in S$.)

Defn. 2 The *universe* $H(\Pi, T, S, \delta)$, usually written as just H, is defined as follows:

$$\begin{cases} H_0 = S \\ H_{i+1} = H_i \cup \{F^K(h_1, \ldots, h_n) \mid h_1, \ldots, h_n \in H_i \text{ and } T(K)=n\} \\ H = \bigcup_{i=1}^{\infty} H_i \end{cases}$$

Defn. 3 A *canonical model* $\mathfrak{A} = \langle A, R, G \rangle$ in the layer $\langle \Pi, T, S, \delta \rangle$ is constructed as follows:

① Define an equivalence relation $\approx$ on H such that if $T(K)=n$ and $h_i \approx h_i'$ for $1 \le i \le n$, then $F^K(h_1, \ldots, h_n) \approx F^K(h_1', \ldots, h_n')$.

② The universe A of $\mathfrak{A}$ is defined to be $S/\approx$.

(3) The functions G of $\mathfrak{A}$ are defined as follows:

If $\tau(K)=0$, then the value of F^K is $\delta(F^K)^{\approx}$.

If $\tau(K)>0$, then the value of $F^K(h_1^{\approx},\ldots,h_n^{\approx})$ is $F^K(h_1,\ldots,h_n)^{\approx}$.

(4) The relations R of $\mathfrak{A}$ may be defined on $S/\approx$ in any manner.

<u>Corollary 4</u>

If $\mathfrak{A}$ is a canonical model of the layer $\langle \Pi, T, S, \delta\rangle$, its cardinality is not greater than $\omega \cup |\beta| \cup |S|$ where β is the domain of T. The collection of canonical models of a given layer have a cardinality, and are a set.

<u>Example 5</u> $\langle \phi; \cdot, ^{-1}, \underline{e}; \{e, a\}; \delta\rangle$ where δ maps $\underline{e}$ into e. This is the layer of groups with one free generator. H contains elements such as $e, a, a^{-1}, e\cdot(a\cdot a^{-1}), a\cdot(a\cdot(a\cdot a))$ etc.

The largest group in this layer is $\mathbb{Z}$. The layer contains the trivial group, and all cyclic groups

Z_n where $n<\omega$.

<u>Theorem 6</u>: If $\mathfrak{A}$ is any model of similarity type $\langle \pi, \tau \rangle$, then for some S, and some δ, there is a canonical model $\mathfrak{H}$ of the layer $\langle \pi, \tau, S, \delta \rangle$ such that $\mathfrak{A} \cong \mathfrak{H}$.

<u>Proof</u>: Take S to be A, and δ to be the $\mathfrak{A}$ interpretation of the 0-ary functions. Define the mapping $f: H \to A$ as follows:

① Since $H_0 = A$, $f|_{H_0}$ is the identity function of A.

② Assuming $f|_{H_i}$ to be defined, define $f|_{H_{i+1}}$ by $f(F^K(h_1, \ldots, h_n))$ being the $\mathfrak{A}$ interpretation of $F^K(f(h_1), \ldots, f(h_n))$.

f is onto since $f|_{H_0}$ is onto.

Define $\approx$ on H by $h_1 \approx h_2$ iff $f(h_1) = f(h_2)$.

$f/\approx : H/\approx \to A$ is 1-1 onto and carries functions defined on $H/\approx$ canonically into the functions of $\mathfrak{A}$.

$\mathfrak{B} = \langle H/\approx, R, G \rangle$ can now be defined by carrying the relations over from $\mathfrak{A}$, and $\mathfrak{A} \cong \mathfrak{B}$.

Throughout this paper, we shall say "model" and mean "canonical model." Theorem 6 justifies this.

<u>Notation</u>: When discussing two models, $\mathfrak{A}, \mathfrak{B}$ of the same layer, it is necessary to distinguish their equivalence relations on H. Thus we use $\approx_{\mathfrak{A}}$ and $\approx_{\mathfrak{B}}$.

<u>Definition 7</u>: Let $\mathfrak{A}, \mathfrak{B}$ be of the layer $\langle \pi, \tau, S, \delta \rangle$. Define $\mathfrak{A} \ll \mathfrak{B}$ iff

① $h_1 \approx_{\mathfrak{B}} h_2$ implies $h_1 \approx_{\mathfrak{A}} h_2$

② If $R^K(h_1^{\approx\mathfrak{B}}, \ldots, h_n^{\approx\mathfrak{B}})$ is true in $\mathfrak{B}$, then $R^K(h_1^{\approx\mathfrak{A}}, \ldots, h_n^{\approx\mathfrak{A}})$ is true in $\mathfrak{A}$.

Note that $\ll$ is neither necessary or sufficient for $\subseteq$. $\mathfrak{A} \subseteq \mathfrak{B}$ "$\mathfrak{A}$ is a substructure of $\mathfrak{B}$"

$\mathfrak{A} \ll \mathfrak{B}$ "$\mathfrak{A}$ is a quotient structure of $\mathfrak{B}$" or "$\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$."

$\ll$ is an partial ordering.

Under certain conditions, $\mathfrak{A} \ll \mathfrak{B}$ means that there exists a $\mathfrak{I} \subseteq \mathfrak{B}$ with $\mathfrak{B}/\mathfrak{I} \cong \mathfrak{A}$.

<u>Theorem 8</u>: Let $\mathfrak{A} \ll \mathfrak{B}$. Then there is a natural homom. $f: \mathfrak{B} \to \mathfrak{A}$.

<u>Proof</u>: Let $f(h^{\approx \mathfrak{B}}) = h^{\approx \mathfrak{A}}$. This is well defined by the definition of $\ll$. It is an immediate consequence of Defn. 7 that functions and relations are also preserved.

Note that:

$$f\left(F^K(h_1 \ldots h_n)^{\approx \mathfrak{B}}\right) = F^K(h_1 \ldots h_n)^{\approx \mathfrak{A}} =$$

$$F^K(h_1^{\approx \mathfrak{A}} \ldots h_n^{\approx \mathfrak{A}}) = F^K(f(h_1^{\approx \mathfrak{B}}) \ldots f(h_1^{\approx \mathfrak{B}}))$$

<u>Theorem 9</u>: Let $\{\mathfrak{A}_i\}_{i \in I}$ be a non-empty totally ordered subset of the models in $\langle \Pi, T, S, \delta \rangle$ under $\ll$. Then there is a model $\mathfrak{J}$ in $\langle \Pi, T, S, \delta \rangle$ which is a l.u.b. for $\{\mathfrak{A}_i\}$.

Proof: Define $h_1 \approx_{\mathfrak{A}} h_2$ iff $h_1 \approx_{\mathfrak{A}_i} h_2$ for all $i \in I$.

Define $R^K(h_1^{\approx\mathfrak{A}}, \ldots, h_n^{\approx\mathfrak{A}})$ to hold iff $R^K(h_1^{\approx\mathfrak{A}_i} \ldots h_n^{\approx\mathfrak{A}_i})$ holds for all $i \in I$.

It is trivial to verify that $\mathfrak{A}_i \ll \mathfrak{A}$ for all $i \in I$, and that this is the minimal construction that will create an upper bound for all $\mathfrak{A}_i$.

Theorem 10: There are maximal elements in the layer $\langle \Pi, T, S, \delta \rangle$, and every element is bounded by one.

Proof: Immediate by theorem 9 and Zorn's lemma.

Notation: Given the layer $\langle \Pi, T, S, \delta \rangle$, we wish to extend the similarity type to include a 0-ary constant to represent each element of H.

For each model $\mathfrak{A}$, there is a model in the extended similarity type. Call this $(\mathfrak{A}, H)$. It is like $(\mathfrak{A}, \underset{\sim}{a})$ but the enumeration is generally redundant. The purpose is to make the extension uniform for all models of a given layer.

The diagram of $(\mathfrak{A}, H)$ is defined as:

$$\{\varphi \mid \varphi \in \Psi^*(\pi, T, H) \cap \Sigma(\pi, T, H) \text{ and } (\mathfrak{A}, H) \models \varphi\}$$

<u>Theorem 11</u>: Let $\{\mathfrak{A}_i\}_{i \in I}$ be a non-empty totally ordered subset of the models of $\langle \pi, T, S, \delta \rangle$. Let σ be a sentence such that $\mathfrak{A}_i \models \sigma$ for all $i \in I$. Let $\mathfrak{A}$ be the lub of $\{\mathfrak{A}_i\}$. Then $\mathfrak{A} \models \sigma$.

<u>Proof</u> Suppose $\mathfrak{A} \not\models \sigma$. Then by compactness, σ is inconsistent with some finite subset $\{T_j\}$ of the diagram of $(\mathfrak{A}, H)$. This would be contradictory if we can show that $\{T_j\}$ are also a subset of the diagram of some $\mathfrak{A}_i$, for then $\mathfrak{A}_i \not\models \sigma$.

Each atomic T_j is in the diagram of all $(\mathfrak{A}_i, H)$.

Each T_j that is the negation of an atomic sentence is in the diagram of some $(\mathfrak{A}_i, H)$. It is also in the diagram of any other $(\mathfrak{A}_i', H)$ if $\mathfrak{A}_i \ll \mathfrak{A}_i'$. Since there are finitely many of them, they are all in the diagram of some $(\mathfrak{A}_i, H)$.

<u>Notation</u>: Let $\langle \Pi, T, S, \delta \rangle$ be a layer, and Γ a theory in $\Sigma(\Pi, T)$. The set of models of the layer that satisfy Γ is $\mathcal{M}(\Gamma, S, \delta)$.

<u>Theorem 12</u>: Let $\{\mathfrak{A}_i\}_{i \in I}$ be non-empty and well ordered in $\mathcal{M}(\Gamma, S, \delta)$. Then the lub $\mathfrak{A}$ of $\{\mathfrak{A}_i\}$ is also in $\mathcal{M}(\Gamma, S, \delta)$.

<u>Proof</u>: Immediate from theorem 11.

<u>Theorem 13</u>: Every element of $\mathcal{M}(\Gamma, S, \delta)$ is bounded by a maximal element also in $\mathcal{M}(\Gamma, S, \delta)$.

<u>Proof</u>: Immediate from Theorem 12 and Zorn's lemma.

<u>Definition 14</u> The $\boxed{\text{free models}}$ of the theory Γ are precisely the maximal elements of all the layers $\mathcal{M}(\Gamma, S, \delta)$.

<u>Example 15</u> The theory of groups.

Sim. type $\langle \phi ; \cdot , ^{-1}, \underline{e} \rangle$

$\Gamma = \{ \forall x\,(x \cdot e \doteq x) \ \ \forall x\,(x \cdot x^{-1} \doteq e) \ \ \forall xyz\,(x \cdot (y \cdot z) \doteq (x \cdot y) \cdot z) \}$

It is sufficient to consider one S of each cardinality and to identify one member of S with $\underline{e}$.

<u>Case I</u>: $S = \{e\}$. Only the trivial group. The free group on no generators.

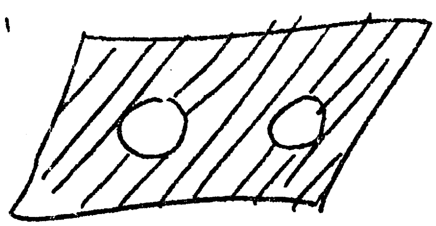
<u>Case II</u>: $S = \{e, a\}$ This layer contains the integers, the cyclic groups of finite order, and the trivial group.

$\mathbb{Z}$ is the unique maximal elt. (free group) of this layer. Every element of H is equivalent to an element of one of the three following forms by the axioms Γ.

(i) $a \cdot a \cdot a \ldots$ n-times
(ii) e
(iii) $a^{-1} \cdot a^{-1} \cdot a^{-1} \ldots$ n-times

Since $\mathbb{Z}$ makes no additional equivalences, it is unique and maximal. Every group in this layer is a homomorphic image of $\mathbb{Z}$.

Case III $S = \{e, a, b\}$. This is the smallest layer to have non-abelian groups in it. Its unique ~~free~~ group is the one with two free generators. It is the fundamental group of:



Example 16: The theory of fields.

Sim. type $\langle 4; +, \cdot, -, ^{-1}, \underline{0}, \underline{1} \rangle$.

Let $S = \{0, 1\}$, $\delta: \begin{matrix} \underline{0} \to 0 \\ \underline{1} \to 1 \end{matrix}$

This layer contains $\mathbb{Q}$ and $\mathbb{Z}_p$ for each p. Every ~~group~~ field in the layer is free. This is not very interesting.

If $S = \{0, 1, X\}$ we get the field of rational functions of the indeterminite X. This is usually denoted $Q(X)$.

<u>Definition 17</u>: A sentence is <u>simple</u> if it has no Boolean connectives, i.e. prenex + atomic formula. A theory is simple if it has a set of simple axioms.

<u>Speculation 18</u>: That simple theories have interesting free models.

<u>Speculation 19</u>: "$\mathfrak{A} \leq \mathfrak{B} \Rightarrow \mathfrak{B}/\mathfrak{A} \cong I$ and $I \ll \mathfrak{A}$." Does the fundamental theorem of groups hold for all simple theories? What is a normal substructure?

<u>Speculation 20</u>: In a simple theory, each layer has a unique free model?

<u>Speculation 21</u>: That simple theories (e.g. groups, rings, semi-groups) have a Homological Algebra.

Speculation 22: That the category of models is Abelian for a simple theory. (See Freyd; Abelian Categories).

Speculation 23: That the concept of free model is equivalent to the categorical concept of projective.

$$\begin{tikzcd}
 & P \arrow[dl, dashed] \arrow[d] & \\
A \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

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