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DIFFERENTIAL PERCEPTRONS

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CONTENTS

INTRODUCTION

As originally proposed, perceptrons were machines that scanned a discrete retina and combined the data gathered in a linear fashion to make decisions about the figure presented on the retina. This paper considers differential perceptrons, which view a continuous retina. Thus, instead of summing the results of predicates, we must now integrate. This involves setting up a predicate space which transforms the typical perceptron sum, $\sum \alpha(\varphi)\, \varphi(f)$,into $\sum_i \int a(p)\, f(p)\, dp$, where f is the figure on the retina, i.e. in the differential case, the figure is viewed as a function on the predicate space. We show that differential perceptrons are equivalent to perceptrons on the class of figures that fit exactly onto a sufficiently small square grid. By investigating predicates of various geometric transformations, we discover that translation and symmetry can be computed in finite order using finite coefficients in both continuous and discrete cases. We also note, that in the perceptron scheme, combining data linearly implies the ability to combine data in a polynomial fashion.

BASIC CONCEPTS

We are only going to consider subsets of the plane that fit an intuitive idea of "nice"; sets that one could draw a picture of.

A figure, f, is a subset of the plane such that:

1) f is compact

2) every point x in f is either an interior point or has an interior point within a distance e, for all $e > 0$.

3) f has only a finite number of components and holes.

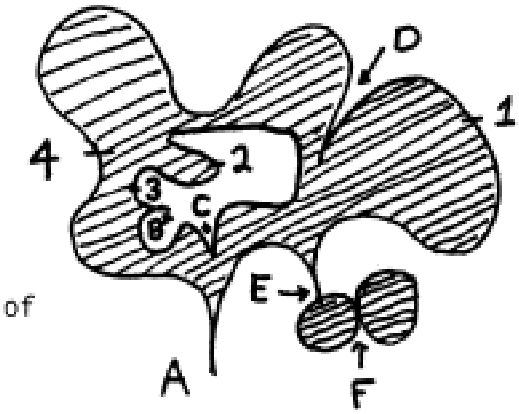
4) for every point x on the boundary of f:

a) there exists $e > 0$ such that curvature is defined at all boundary points within a punctured e-ball about x.

b) either curvature is defined at x, denoted C_x, or the limit of the tangents from either side makes an angle at x, denoted A_x.

Let F be the set of all figures.

Points on a straight line segment on the boundary are defined to have curvature equal to zero and no angle defined. The magnitude of A_x is the radian measure of the amount one turns upon passing the vertex. For instance, the angle at the vertex of a regular n-gon is $2\pi/n$, the exterior angle rather than the interior angle. $|A_x|$ will always lie in $(0,\pi)$ and the sign can be determined as follows: if one moves around the edge, keeping the figure to one's right, and the interior of the angle lies to one's right then the angle is positive. Similarly, if one moves about the edge keeping the figure to one's right and one is moving clockwise with respect to the center of curvature then the curvature is positive.



For example, in the adjacent figure the curvature at 1 and 2 is positive; at 3 and 4 it is negative. The angles at A and B are positive; the angles at C, D, and E are negative. The angle at F is the limit of tangents of tangent circles, it is $-\pi$.

For the remainder of this paper by <u>predicate</u> we mean a function $F \rightarrow \{0,1\}$. A <u>predicate of finite index</u>, p, is one whose value depends only on whether each point of some fixed finite subset of the retina, X, satisfies a given local property with respect to the retina. X is the <u>support</u> of p, denoted $S(p) = X$. The <u>index</u> of p is $|X|$.

The simplest example is the index 1 predicate p^x whose support is $\{x\}$ and $p^x(f) = 1$ iff $x \in f$. Let $(X_1, X_2, X_3, X_4) = \vec{X} \in (\mathbb{R}^2)^4$. We can define a predicate $p^{\vec{X}}$ by $p^{\vec{X}}(f) = 1$ iff $X_1 \in f$, $X_2 \notin f$, X_3 is an interior point of f, and X_4 is a boundary point of f where curvature is defined and equal to $\pi/2$. The index of $p^{\vec{X}}$ is the number of distinct points amoung X_1, X_2, X_3, X_4 and $S(p^{\vec{X}})$ is those points.

Most sets of predicates of finite index may be topologized so that they are at least a subset of a topological group X that has an invariant integral defined upon it. If we view a figure f as a function on this space via $f(p) = p(f)$ then in most interesting cases it will be clear that f is integrable. More generally, we can then write $\int_X a(p) f(p)\, dp$ where $a: X \rightarrow \mathbb{R}$ is a reasonable function.

Let p^x be the index 1 predicate as above. Let $K = \bigcup_{x \in \mathbb{R}^2} p^x$. Then we may topologize K with the topology of the plane, i.e. the predicates correspond bijectively with their supports. Then the integral over K is the Lebesgue integral. Alternately we may topologize K with the discrete topology resulting in the counting integral $\sum_{p \in K} a(p) f(p)$.

Suppose J is the set of all index 1 predicates p such that $p(f) = 1$ iff $S(p)$ is a boundary point of f. We can topologize J so that it is homeomorphic to $\mathbb{R}^2$ and $\int_J f(p) dp$ is equivalent to $\int_{\mathbb{R}^2} C(x, y)\, dxdy$ where C is the characteristic function of the boundary of f. Unfortunately, the boundary is a set of measure zero in the plane. We would, however, like to be able to associate a measure with it, the length of the boundary, which is defined for $f \in F$. Suppose then that for each $x = (x_1 \ldots\ldots x_n) \in (\mathbb{R}^2)^n$ there is a predicate p^x, such that $p^x(f) = 1$ iff each x_i is a boundary point of f and perhaps satisfies some other property as well. Then this set of predicates, call it L, can be topologized so that it is homeomorphic to $(\mathbb{R}^2)^n$. We will use the Lebesgue taken over the surface in $\mathbb{R}^{2n}$ corresponding to the boundary of f, i.e. $(x_1 \ldots\ldots x_n) \in R^{2n}$ is on this surface iff each x_i is a boundary point of f. This integral is normalized so that the "area" of the surface is the length of the boundary of f raised to the power n.

We are now ready to fit these integrals into the general perceptron scheme: a linear sum with a threshold. The idea is that we have a set of predicate spaces $\{K_\alpha\}_{\alpha \in \Lambda}$ and an integral over each, $\int_{K_\alpha} a(p) f(p) dp$ where the integral may be any of the types discussed, and we now wish to sum these integrals in some manner. Note that it is quite possible for there to be a topological structure on this set of predicate spaces, for instance if I_r is the space of all index 1 predicates where $p \in I_r$ is such that $p(f) = 1$ iff $S(p)$ is a boundary point of f and curvature is defined there and equal to r then the set $\{I_r\}_{r \in \mathbb{R}}$ may be given the topology of the line and so to sum integrals of the form $\int_{I_r} a(p)\, f(p)\, d\, p$ we may write $\int_{\mathbb{R}} \int_{I_r} a(p) f(p) dpdr$.

In general, we may identify the set of predicate spaces, $\{K_\alpha\}_{\alpha\in\Lambda}$, that we wish to sum over in a differential perseptron, with a subspace of a topological group that has an invariant integral. Indeed, the discrete topology admits any group structure as a topological group and any set is a subset of the free group generated upon it; therefore, we may write a percetpron as $\psi(f) = \lceil \int_\Lambda \int_{K_\alpha} a(p) f(p)\, dp \leq \theta \rceil$. The order of ψ is the maximun of the indices of predicates in any of the K_α.

II EXAMPLES

To illustrate the concepts of the previous section this section consists of a number of examples of differential perceptrons. The first three examples demonstrate the usefulness of topologizing predicate spaces with the discrete topology so that integrating reduces to counting.

Example 1: $\psi_{\text{convex}} = \ulcorner$The figure on the retina is convex$\urcorner$.

Note that a figure is convex if and only if there exists a pair of points belonging to the figure such that the midpoint of the line segment joining them is not in the figure. Let p^x be the index 1 predicate such that $f(p^x) = 1$ if and only if $x \in f$. Let $\mathcal{U}$ be the set of all predicates $p^{x_1} \wedge \neg p^{x_2} \wedge p^{x_3}$ as x_1, x_2, and x_3 run independently over the retina. For $p \in \mathcal{U}$ let $a(p) = 1$ if x_2 is the midpoint of $\overline{x_1 x_3}$ and zero otherwise. Then giving the discrete topology we can write the order 3 perceptron:

$$\psi_{\text{convex}}(f) = \ulcorner \sum_{\mathcal{U}} a(p) f(p) \leq 0 \urcorner .$$

Example 2: $\psi_{\text{reg. poly.}} = \ulcorner$The figure on the retina is a regular polygon$\urcorner$.

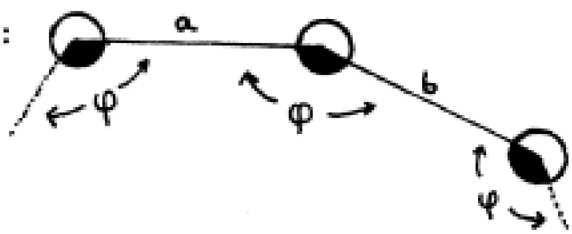
A figure will not be a regular polygon if and only if it satisfies at least one of the following properties:

1. It is not convex.
2. It has nonzero curvature at some boundary point.
3. It has two unequal angles.
4. It has three adjacent vertices a, b, c, such that the length of ab does not equal the length of bc.

Let V be the set of index one predicates where for $p \in V$, $S(p) = \{x\}$ and $f(p) = 1$ if and only iff x is boundary point of f where curvature is defined and not equal to zero.

Let W be the set of all predicates $p = \lceil$angle is defined at x and at y and is unequal at those two places$\rceil$ as x and y run independently over the retina.

Let X be the set of all predicates which recognize the following 3 point configuration:



where $0 < \varphi < \pi$ and $a \neq b$.

Let U be as previously then, giving V,W, and X discrete topologies:

$$\Psi^{(f)}_{\text{reg. poly}} = \lceil \sum_{U} a(p)f(p) + \sum_{V} f(p) + \sum_{W} f(p) + \sum_{X} f(p) \leq 0 \rceil .$$

If we wish to recognize only regular n-gons then let Y be the space, with discrete topology, of all index 1 predicates that output 1 when their support lies on a boundary point which is the vertex of some positure angle other than $2\pi/n$. Then adding $\sum_{Y} f(p)$ to the linear sum for Ψreg. poly. yields Ψ reg. n-gon.

Example 3: Circles

A figure is not a circle if and only if it satisfies one of the following properties:

1. The figure is not convex.
2. There is a boundary point where a nonzero angle occurs.
3. There exist two boundary points x,y, such that $C_x \neq C_y$.

Thus, let Z be the space of all index 1 predicates such that for $p \in Z$, $f(p) = 1$ if and only if $S(p)$ is a boundary point of f where a non-zero angle is defined.

Let S be the space of all predicates $p = \ulcorner$ x and y are boundary points and $Cx \neq Cy \urcorner$ as x and y range over the retina.

Then, with U as before , and with discrete topologies all around:

$$\psi_{circle} = \ulcorner \sum_U a(p) f(p) + \sum_Z f(p) + \sum_S f(p) \leq 0 \urcorner .$$

If we wish only to recognize circles with radius r then let T be set, with discrete topology, of all predicates $p = \ulcorner$ x is a boundary point where curvature is defined and not equal to $1/r \urcorner$. Then adding in $\sum_T f(p)$ to the sum for ψcircle yields ψcircle of radius r.

Example 4: total curvature

Suppose Cx and Ax are the classes of index 1 predicates at x such that all for $r \in \mathbb{R}$ and $s \in [-\pi, \pi]$ there exists $p \in Cx$, $q \in Ax$ such that $p = \ulcorner$ x is a boundary point where curvature is defined and equal to $r \urcorner$ and $q = \ulcorner$ x is a boundary point where angle is defined and equal to $s \urcorner$. Let $a(p) = r$ and $a(q) = s$ then giving Cx and Ax the discrete topology and summing:

$\sum_{Cx} f(p)$ = The curvature at x, if defined; 0 otherwise.

$\sum_{Ax} f(p)$ = The angle at x, if defined; 0 otherwise.

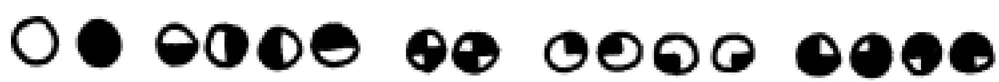
Thus if the outside integral in $\int \left(\sum_{Ax} f(p) + \sum_{Cx} f(p) \right) dx$ is the boundary integral discussed previously then the entire integral is the total curvature of f. This determines that the Euler number (components-holes) of the figure is less than some threshold.

III. RESTRICTED EQUIVALENCE OF DISCRETE AND DIFFERENTIAL PERCEPTRONS

In this section we show a restricted equivalence between discrete perceptrons and differential perceptrons.

If we superimpose a square lattice on the retina we may consider it as the retina of a discrete perceptron. Figures that fit neatly into this grid, i.e. figures that the discrete and differential perceptron have in common will be referred to as squared figures. We will show that discrete perceptrons and differential perceptrons are equivalent when restricted to squared figures. The only discrepency arising in this restriction is that corner connected figures are locally connected at that corner in the standard topology of the plane whereas, by the convention stated in Perceptrons, on the discrete retina there is no connection there. We will reverse this convention and consider such a figure to be connected and consider its complement to be locally disconnected at that point.

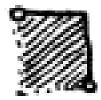
If one looks at a small neighborhood of a point on the retina that has a squared figure on it one may see only one of the following:



Thus any index 1 predicate satisfied by such a figure is equivalent to an index 1 predicate which recognizes one or more of the above.

Thus if a differential perceptron is applied only to squared figures we may eliminate from all sums the predicates which are never satisfied by a squared figure. Then if that perceptron is of finite order it may be cast in the form $\Psi(f) = \lceil \sum_{i=1}^{m} \int_{I_i} a(p)f(p)\, dp < \theta \rceil$. We will show that if we put a grid on the retina and consider it as the retina of a discrete perceptron then we can construct a class of predicates over this discrete retina, for each i, such that the summation of these classes yields the same value as $\int_{I_i} a(p)\ f(p)dp$.

In the following discussion we shall assume that such a sum, Ψ, with a fixed threshold, θ, is given. We will show how to construct a sum of predicates over the retina, now viewed as a discrete retina, which is identical in value to Ψ.

Visually, an e-grid divides the plane into boxes; we can make them disjoint by including with each box only its upper and right hand edges and not the end corners of these edges, i.e.  is an entire box.

Consider the retina with a grid on it. If we let "subsets" be subsets (of the retina) which are composed entirely of boxes then for each integer we can order the "subsets" consisting of exactly m boxes. Let B^k_m be the K^{th} such "subset" having m boxes. This gives us a partition of any predicate space I:

$$I^k_m = \{ p \in I \mid S(p) \subset B^k_m \text{ and for all } j \text{ and all } l < m \ S(p) \not\subset B^j_l \}$$

In other words, I^k_m is all $p \in I$ such that $S(p) \subset B^k_m$ and at least one point of S(p) is in each box in B^k_m. If (all) $p \in I$ has index m then for $m > n$, $I^k_m = \emptyset$. Therefore, $\int_I a(p)f(p)dp = \sum_{m,k} \int_{I^k_m} a(p)f(p)dp$.

Each B^k_m has a finite number of "subsets," and these can be numbered. Let ${}_jD^k_m$ be the j^{th} such "subset".* For p I^k_m there is at most one "subset" ${}_{j_o}D^k_m$ such that $p({}_{j_o}D^k_m) = 1$; for all others $p({}_jD^k_m) = 0$. Thus regardless of what a squared figure f looks like outside of B^k_m, $p(f) = 1$ if and only if $f \cap B^k_m = {}_{j_o}D^k_m$. Now I^k_m can be paritioned into $(\bigcup_j {}_jJ^k_m) \cup J$ where $p \in {}_jJ^k_m$ if and only if $p({}_jD^k_m) = 1$ and J is the set of all predicates in

* We will also use ${}_jD^k_m$ to represent the (squared) figure consisting only of the "subset" ${}_jD^k_m$.

I_m^k that are never satisfied by a squared figure. Thus for $j_1 \neq j_2$, ${}_{j_1}J_m^k \cap {}_{j_2}J_m^k = \phi$ and $\int_{I_m^k} a(p)f(p)dp = \sum_j \int_{{}_jJ_m^k} a(p)f(p)dp$.

At this point we must consider what type of integral we are dealing with in the expression $\int_I a(p)f(p)dp$ (which represents one of the summands in $\Psi = \left\lceil \sum_{i=1}^{m} \int_{I_i} a(p)f(p)dp < \theta \right\rceil$). If this integral is the Lebesgue integral or the Lebesgue integral over the boundary, as described in Basic Concepts then it will be finite over any figure since all figures are compact. If, however, I has the discrete topology and the integral is the counting integral then it may be infinite. Thus we define predicates over the retina with the superimposed grid, now viewed as the retina of a discrete perceptron, in two ways. If the above integral is the counting integral and the entire sum is finite when taken over any figure, for instance if its predicates look for corner features, or if the integral is of the first two types, then let ${}_jP_m^k$ be a predicate over the (now) discrete retina with coefficient $A({}_jP_m^k)$ by $f({}_jP_m^k) = \left\lceil \int_{{}_jJ_m^k} a(p)f(p) \neq 0 \right\rceil$ and $A({}_jP_m^k) = \int_{{}_jJ_m} a(p)\,{}_jD_m^k(p)dp$, this latter coefficient always being finite. If, however, there are figures for which $\sum_I a(p)f(p)$ is infinite, for instance if its predicates look for internal points of f, then we note that the integral must be either infinite or zero. Thus we let ${}_jP_m^k = \left\lceil \int_{{}_jJ_m^k} a(p)f(p) \neq 0 \right\rceil$ as before, but since we cannot have infinite coefficients in the discrete perceptron scheme, we instead let $A({}_jP_m^k) = \pm 1$, depending on whether $\sum_{{}_jJ_m^k} a(p)f(p) = \pm\infty$ and in the linear sum we are constructing, allow $A({}_jP_m^k)$ to appear infinitly often so that the effect is the same. Thus $\int_I a(p)f(p)dp = \sum_{m,k,j} A({}_jP_m^k)f({}_jP_m^k)$.

We can now write a discrete perceptron equivalent to by applying the above to each I_i in $\sum_{i=1}^{m}\int_{I_i} a(p)f(p)dp$. Note that this new perceptron has order less than or equal to the order of ψ.

Now we will show that if we have a discrete perceptron defined upon the retina that we may construct a differential perceptron that is equivalent when restricted to squared figures. Let f be a (squared) figure on the discrete retina. Let the area of each box in the grid be E. If we have a linear sum for a discrete perceptron, $\psi(f) = \lceil \sum a(Q)f(Q) \rceil$ we may assume that is in positive normal form.

Let Q_A be a mask (in the discrete sense), Q_A (f) = 1 if and only if $f \subset A$, for some fixed "subset" A of the retina. Then $A = \bigcup_{i=1} A_i$ where A_i are boxes. Define M_A to be the class of predicates (working on the continuous retina) containing all predicates of index n, with $S(p) \subset A$ and $S(p) \not\subset B$ for any proper "subset" B of A, i.e. there is one point of S(p) in each A_i and for any n-tuple of points $x_1, \ldots, x_n$* and p (f) = 1 if and only if $x_i \in f$, for i=1, 2..., n. Furthermore, since a figure occupies all of A_i or none of it, for $p \in M_A$, p(f) = 1 if and only if $A \subset f$. Then, $\int_{M_A} f(p)dp = E^n$ if $A \subset f$ and 0 otherwise. So Q_A (f) = $1/E^n \int_{M_A} f(p)dp$. Therefore we can duplicate any sum over the discrete retina as the sum of integrals of the above type.

What we have proved is that for any linear sum of order n over the discrete retina, i.e. any discrete perceptron, we have an equivalent linear sum of order n over the continuous retina, which at least works for squared figures. Conversely, if we have a linear sum over the continuous retina of order n which works for squared figures then there is an equivalent linear sum of order n over the discrete retina. In particular, anything which

* with $x_i \in A_i$ There is one $p \in M_A$ with $S(p) = \{x_1 \ldots x_n\}$

cannot be calculated as a linear sum of finite order with respect to the discrete retina, with our reversed convention concerning corner connectedness, cannot be done in general with respect to the continuous retina in finite order. Since the proof in <u>Perceptrons</u> that connected has no representation as a linear sum with finite order does not involve this convention the result follows: there is no linear sum of finite order over the continuous retina which determines ψ connected.

IV GEOMETRIC TRANSFORMATIONS

In this section we discuss some interesting differential perceptrons. We will consider the retina to be divided into two half planes, A the left half, B the right. Given a figure in each half we are interested in having a perceptron recognize whether or not they differ by a geometric transformation; e.g. translation, rotation, reflection, and combinations thereof.

The space of predicates we will use, K, is the space of index 3 predicates such that for $p \epsilon K$, $S(p) = \{x_1, x_2, x_3\}$ and $f(p) = 1$ if and only if x_1, x_2 and x_3 are interior points of f. We think of K as partitioned into equivalence classes under translation; to be precise, the equivalence class of $p \epsilon K$ is all $q \epsilon K$ such that $S(q) = tS(p)$ where $tS(p)$ is a translation of $S(p)$. Thus each equivalence class consists of all predicates whose support is some fixed configuration of three (not necessarily distinct) points. Given a configuration C, K_C will denote the equivalence class of all predicates in K whose support is C. The topology of a class K_C is that of R^2 as is easily seen from the bijective map from the center of gravity of the support of each predicate in K_C to the plane.

Given a space of predicates S, S^A and S^B will denote the subspaces of S with support lying entirely in A and B respectfully.

Suppose two figures f and g lie in A and B respectfully. Given two configurations C_1 and C_2, not necessarily distinct, define $K^{AB}_{C_1C_2} = K^A_{C_1} \times K^B_{C_2}$. $p \epsilon K^{AB}_{C_1C_2}$ is of the form $p_1 \times p_2$, $p_1 \epsilon K^A_{C_1}$, $p_2 \epsilon K^B_{C_2}$ and $S(p) = S(p_1) \cup S(p_2)$ and $f(p) = f(p_1)f(p_2)$. That is $K^{AB}_{C_1C_2}$ is a space of index 6 predicates and is a four dimensional subspace of R^4. If we now look at the retina as a whole it containsthe figure $f \cup g$, denoted fg. If $p \epsilon K^{AB}_{C_1C_2}$ then $fg(p) = fg(p_1 \times p_2) =$

$f(p_1)g(p_2)$. Therefore by the Fubini theorem:

$$\int_{K^{AB}_{C_1C_2}} fg(p)dp = \int_{K^A_{C_1}} \left(\int_{K^B_{C_2}} f(p_1)g(p_2)dp_2\right)dp_1 = \left(\int_{K^A_{C_1}} f(p_1)dp_1\right) \left(\int_{K^B_{C_2}} g(p_2)dp_2\right).$$

Similarly, if we define $K^A_{C_1C_2} = K^A_{C_1} \times K^A_{C_2}$ and f lies in A then

$f(p_1 \times p_2) = f(p_1)f(p_2)$ so:

$$\int_{K^A_{C_1C_2}} f(p)dp = \int_{K^A_{C_1}}\left(\int_{K^A_{C_2}} f(p_1)f(p_2)dp_2\right)dp_1 = \left(\int_{K^A_{C_1}} f(p_1)dp_1\right)\left(\int_{K^B_{C_2}} f(p_2)dp_2\right)$$

If $C=C_1=C_2$ then we write $K^A_{C_1C_2} = K^A_{C^2}$, $K^{AB}_{C_1C_2} = K^{AB}_{C^2}$ and $K^B_{C_1C_2} = K^B_{C^2}$.

Now, if f and g are figures in A and B respectfully; $\left(\int_{K^A_{C_1}} f(p)dp - \int_{K^B_{C_2}} g(p)dp\right)^2 =$

$$\left(\int_{K^A_{C_1}} f(p)dp\right)^2 - 2\left(\int_{K^A_{C_1}} f(p)dp\right)\left(\int_{K^B_{C_2}} g(p)dp\right) + \left(\int_{K^B_{C_2}} g(p)dp\right)^2 =$$

$$\int_{K^A_{C_1^2}} f(p)dp - 2\int_{K^{AB}_{C_1C_2}} fg(p)dp + \int_{K^B_{C_2^2}} g(p)dp.$$

Futhermore, since the support of all predicates in $K^A_{C_1^2}$ have their support in A, $\int_{K^A_{C_1^2}} fg(p)dp = \int_{K^A_{C_1^2}} f(p)dp$ and similarly $\int_{K^B_{C_2^2}} fg(p)dp = \int_{K^B_{C_2^2}} g(p)dp$.

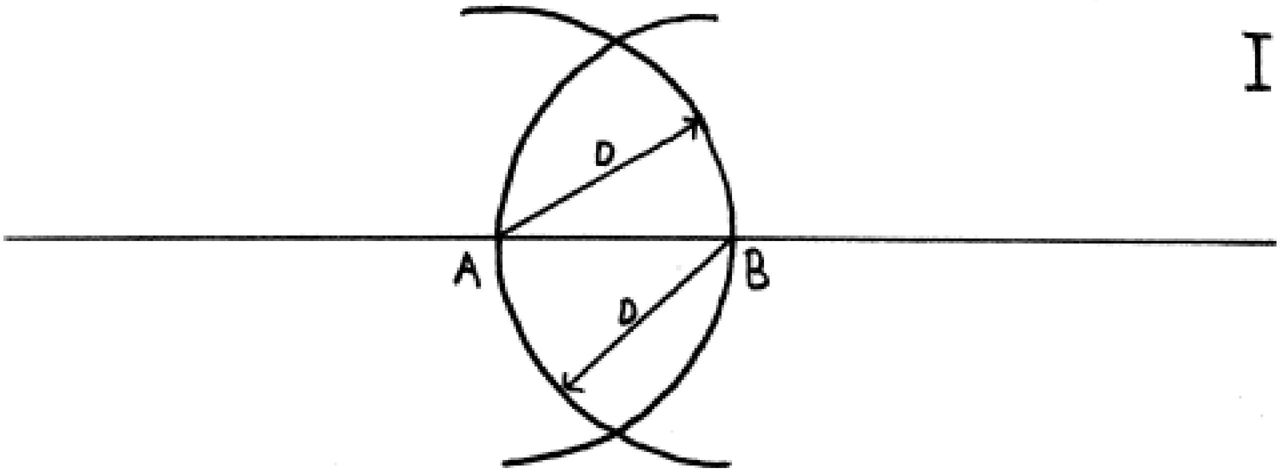
Clearly any finite degree polynomial combination of integrals can be formed in this manner, that is, the concept of finite order linear sums over arbitrary predicate spaces automatically includes finite degree polynomial sums. Thus if we write a polynomial sum as above we will mean the equivalent linear sum over the appropriate spaces.

For a figure f, its internal spectra is the function V:{all three point configurations}→ R^1 where $V(C)= \int_{K_c} f(p)dp$. By the nature of figures if for some $p\epsilon K_C$ f(p)=1 then V(C)>0.

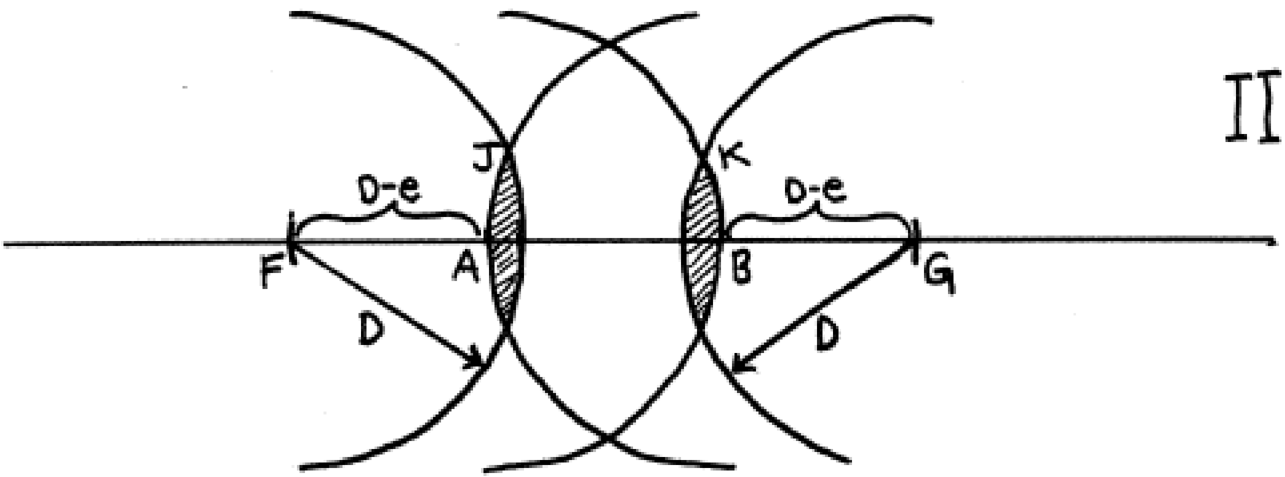
We will prove the following result: Every figure is characterized up to translation by its internal spectra. However, first we need the following lemma from geometry: If a trapezoid has bases of length D and D-e, for some e, 0<e<D, then in order that both diagonals have length less than or equal to D it is necessary that the endpoints of the shorter base each lie within a ball of radius $(De)^{1/2}$ about the associated endpoints of the longer base.

To see this constructed two circles as in I on the following page; $\overline{AB}$ will be the longer base of the trapezoid. To get II mark off distances D-e from A and B to get F and G and construct circles of radius D at F and G. The circle at G and the circle at B have their centers D-e apart, the same is true for the circles at F and A. Thus it is necessary that the endpoints of the shorter base lie in the shaded regions. We would like to calculate the distance $|\overline{AJ}| = |\overline{BK}|$. Since all the circles have the same radius notice that the vertical lines in III bisect the segments of AB lying in the shaded regions. By construction each of these segments has length e. Two applications of the Pythagorean Theorem yields $|\overline{AJ}|= |\overline{BK}| = (DE)^{1/2}$.

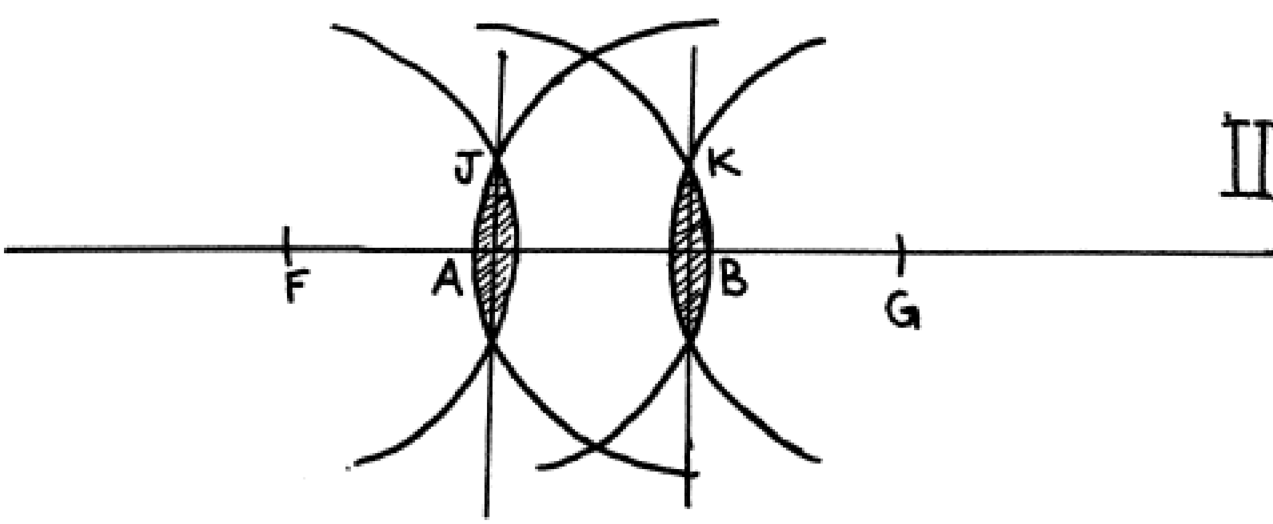
Suppose two figures are translates. Then it is clear that their internal spectra are identical. Conversly, suppose two figures f and g are not translates. Then consider their three vector spectra (as in Perceptrons). We know there are some vectors of maximal length D in f and length E in g; (these have their endpoints on edge points of the figures); but given one



I



II



III

such vector $\vec{D}$ in f (or $\vec{E}$ in g) there are no maximal vectors parallel to it. Choose a maximal vector $\vec{D}$ in f. There are three possibilities:

1) Maximal vectors in g have length $E \neq D$. Suppose, without loss of generality, that $D-E=d>0$. Then there are two interior points of f separated by a distance $D-e$ for some e, $0<e<d$. Pick a third interior point anywhere in f. Call the configuration formed by these three points C. Since no translate of C fits into g we have: $\int_{K_C} f(p)dp>0$ and $\int_{K_C} g(p)dp=0$.

2) Maximal vectors in g have length D but there is no such vector parallel to $\vec{D}$. Hence all pairs of interior points in g lying on a line segment parallel to $\vec{D}$ are separated by a distance less than $D-d$ for some $d>0$. Pick two interior points in f separated by a distance $D-e$ for some e, $0<e<d$ and pick any third point in f. Call the configuration formed by these three points C. Then: $\int_{K_C} f(p)dp>0$ and $\int_{K_C} g(p)dp = o$.

3) g has a maximal vector of length D parallel to $\vec{D}$. Since by assumption f and g are not translates, if we place the figures on top of one another with these maximal vectors coinciding then at least one of f-g and g-f is nonempty, say f-g is. Furthermore, from the definition to figures it follows that f-g has a nonempty interior. Pick an interior point x of f-g; there is a $d>0$ such that a d-ball about x is a subset of f-g. Now pick e, $0<e<d$, such that $2(De)^{1/2}<d$ and such that there are two interior points of f lying on a line segment parallel to D separated by a distance $D-e$. Consider the configuration C formed by these two points and x. By the preceeding lemma and the fact that D is maximal, translates of this configuration do not fit into f

unless the two points separated by the distance D-e are within a distance $(De)^{1/2}$ of the endpoints of D. But then the point in C corresponding to x still lies in the d-ball about x, i.e. it still lies in f-g. Therefore:

$$\int_{K_C} f(p)dp > 0 \qquad \text{and } \int_{K_C} g(p)dp = 0.$$

Since the three vector spectra characterizes a figure in n-space, in particular, since a maximal vector in an n-space figure is unique, the above proof shows that the internal spectra of a figure in n-space; where configurations consist of any three points in n-space; characterizes that figure up to translation (in n-space).

Thus, given a figure f in A and g in B, they are translates if and only if:

$$\int_{K_C^A} f(p)dp = \int_{K_C^B} g(p)dp \text{ for all } C \qquad \text{i.e. if and only if:} \qquad \sum_C \Big(\int_{K_C^A} f(p)dp - \int_{K_C^B} g(p)dp\Big)^2 = 0$$

Therefore:

$$\Psi_{\substack{\text{f and g =}\\ \text{are translates}}} \left\lceil \sum_C \Big(\int_{K_C^A} f(p)dp - \int_{K_C^B} g(p)dp\Big)^2 \leqslant 0 \right\rceil =$$

$$\left\lceil \sum_C \Big(\int_{K_{C^2}^A} fg(p)dp - 2 \int_{K_{C^2}^{AB}} fg(p)dp + \int_{K_{C^2}^B} fg(p)dp\Big) \leqslant 0 \right\rceil$$

This is a perceptron of order 6. Notice the absolute value of the coefficients of any predicate in the sum is less than or equal to two.

Given a configuration C, let $\bar{C}$ be the configuration gotten by reflecting C through the line separating A and B. Given two figures, f in A and g in B, f is a reflection of g through the line separating A and B, up to translation, if and only if:

$\int_{K_C^A} f(p)dp = \int_{K_{\overline{C}}^B} g(p)dp$ for all C.

The equality is clear if f is a reflection of g. Coversly, if f is not a reflection of g then the figure $\overline{g}$ which is g reflected through the line separating A and B translated back into B is not a translate of f. It is obvious that : $\int_{K_C^B} g(p)dp = \int_{K_{\overline{C}}^B} \overline{g}(p)dp \; \forall C.$

But, since f and $\overline{g}$ are not translates:

$$\int_{K_C^A} f(p)dp \neq \int_{K_C^B} \overline{g}(p)dp \text{ for some C.}$$

So $\int_{K_C^A} f(p)dp \neq \int_{K_{\overline{C}}^B} g(p)dp$ for some C.

Thus: $\Psi_{\substack{\text{f is a}\\ \text{reflection}\\ \text{of g}}} = \left\lceil \sum_C \left(\int_{K_C^A} f(p)dp - \int_{K_{\overline{C}}^B} g(p)dp \right)^2 \leqslant 0 \right\rceil$

This idea can be used in another situation. For the moment no longer think of the retina as divided. For an orientation r, and a configuration C, let $\overline{C}$ be the configuration gotten by reflecting C through a line of orientation r. Let f be a figure. Then f has a line of symmetry of orientation r iff

$\int_{K_C} f(p)dp = \int_{K_{\overline{C}}} f(p)dp$ for all C.

i.e. $\Psi_{\substack{\text{There exists a}\\ \text{line of symmetry}\\ \text{of orientation r}}} = \left\lceil \sum_C \left(\int_{K_C} f(p)dp - \int_{K_{\overline{C}}} f(p)dp \right)^2 \leqslant 0 \right\rceil$

Sums of this type for different orientations may be combined to determine whether or not f has multilateral symmetry.

We note that the same scheme, essentially squaring the results of the predicate computation, applies in the obvious manner to discrete perceptrons. Thus symmetry and translation can easily be computed in the discrete case.

We can only conjecture about computing $\Psi_{rotation}$ because as of yet we have been unable to prove the following: figures are determined, up to rotation, by their 3-point spectra (i.e. non-oriented 3-vector spectra). Given this fact, $\Psi_{rotation}$ is quite simple. Let f and g be figures in A and B respectively. Let R_C be the space obtained by operating on K_C with the group of rotations. In other words, R_C consists of all predicates whose supports are rotations and translations of the predicates in K_C. Then, in our notation, 'figures are determined, up to rotation, by their 3 point spectra' looks like $\sum_C (\int_{R_C^A} f(p) - \int_{R_C^B} g(p))^2 = 0$ if and only if f is a rotate (and translate) of g. Clearly, in that case:

$$\Psi_{rotation} = \left\lceil \sum_C \left(\int_{R_C^A} f(p) - \int_{R_C^B} g(p) \right)^2 \leq 0 \right\rceil$$

While we cannot prove the above, we can prove a weaker result that depends on the order of the group of rotations. It will be shown that $\Psi_{rotation}$ can be determined with order less than or equal to $2(|R| + 2)$ where $|R|$ is the order of the group of rotations, R. This means that $\Psi_{rotation}$ can be done in order 12 on the discrete (square) retina since the largest rotation group that can be considered in the discrete case is R = {0, 90, 180, 270} .

The proof is as follows:

Let R be a finite group of rotations of order n with generator r. Let f be a figure in A and g be a figure in B. We will show that if f is not a rotate (i.e. R rotate) of g, there exists a configuration of order n + 2 that fits in f, but does not fit in g under any rotation.

Thus suppose that f and g are not R-rotates. We may suppose that the lengths of maximal vectors appearing in f and g are the same, say D. Pick a maximal vector $\vec{D}$ in f. We may assume that for at least one R-rotation of g there is a maximal vector parallel to $\vec{D}$ since otherwise there is obviously a three point configuration that fits into f but not into g under any R-rotation. For each i such that $r^i g$ has a maximal vector, $\vec{D}_i$, parallel to $\vec{D}$ we may place $r^i g$ on top of f so that $\vec{D}_i$ and $\vec{D}$ coincide and we may assume that $f - r^i g \neq \emptyset$. For such i let x_i be an interior point of $f - r^i g$. Pick $d > 0$ such that a d-ball about x_i lies entirely in $f - r^i g$ when they are lined up as above, for all i. Pick $e \leq d$ such that $2(De)^{1/2} < d$ and such that there are two interior points of f, y and z, separated by a distance D-e and lying on a line segment parallel to $\vec{D}$. Let C be the configuration of n+2 points or less consisting of the x_i's, y and z. By construction and the lemma about trapezoids no R-rotate of C fits into g. Let $\mathcal{C}$ be the set of all configurations having n+2 points or less. Then:

$$\Psi_{R\text{-rotate}} = \left\lceil \sum_{C \in \mathcal{C}} \left(\int_{R_C^A} f(p)\,dp - \int_{R_C^B} g(p)\,dp \right)^2 \leq 0 \right\rceil .$$