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ANY DIMENSIONAL RECONSTRUCTION FROM HYPERPLANAR PROJECTIONS

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Abstract. In this paper we examine the reconstruction of functions of any dimension from hyperplanar projections. This is a generalization of a problem that has generated much interest recently, especially in the field of medical imaging. Computed Axial Tomography (CAT) and Nuclear Magnetic Resonance (NMR) are two medical techniques that fall in this framework. CAT scans measure the x-ray density along lines through the body, while NMR scans measure the hydrogen density along planes through the body.

Here we will examine reconstruction methods that involve backprojecting the projection data and summing this over the entire region of interest. There are two methods for doing this. One method is to filter the projection data first, and then backproject this filtered data and sum over all projection directions. The other method is to backproject and sum the projection data first, and then filter. The two methods are mathematically equivalent, producing very similar equations.

We will derive the reconstruction formulas for both methods for any number of dimensions. We will examine the cases of two and three dimensions, since these are the only ones encountered in practice. The equations are very different for these cases. In general, the equations are very different for even and odd dimensionality. We will discuss why this is so, and show that the equations for even and odd dimensionality are related by the Hilbert Transform.

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1. Introduction

In this paper we will examine the reconstruction of functions of any dimension from hyperplanar projections. This is a generalization of a problem that has generated much interest recently, especially in the field of medical imaging. Computed Axial Tomography (CAT) and Nuclear Magnetic Resonance (NMR) are two medical techniques that fall in this framework.

CAT scans measure the x-ray density along lines through the body. Reconstruction of the density distribution results in a 2-dimensional image from line projections. The line projections are degenerate hyperplanar projections for the 2-dimensional case.

NMR scans measure the hydrogen density along planes through the body. Reconstruction results in a 3-dimensional density distribution. The reconstructed distribution is usually sliced along an arbitrary plane to produce a 2-dimensional image for display purposes.

Here we will examine reconstruction methods that involve backprojecting the projection data and summing this over the entire region of interest. Such methods produce a smeared image, so it is necessary to undo this smearing. There are two methods for doing this. One method, known as the rho-filtered layergram, is to filter the projection data first, and then backproject this filtered data and sum over all projection directions. The other approach is to backproject and sum the projection data first, and then filter. The two methods are mathematically equivalent, producing very similar equations. They differ in that in the first case the filtering need only be performed in one dimension, while in the second case, filtering must be performed in N dimensions. One dimensional filtering is generally easier to implement.

We will derive the reconstruction formulas for both methods for any number of dimensions. We will examine the cases of two and three dimensions, since these are the only ones encountered in practice. The equations are very different for these cases. In general, the equations are very different for even and odd dimensionality. We will discuss why this is so, and show that the equations for even and odd dimensionality are related by the Hilbert Transform.

2. Convolution Followed by Backprojection and Summation

Following (Louis and Natterer 1983), we first introduce some notation. Let R^N be the N-dimensional real space, and let S^{N-1} be the set of directions in R^N. S^{N-1} is formed from all unit vectors in R^N.

$$S^{N-1} = \{x \in R^N \text{ such that } |x| = 1\} \tag{2.1}$$

Let us consider the following problem. The density of the object under study will be denoted by $f(\mathbf{x})$, $\mathbf{x} \in R^N$. We are given projections $p(s, \mathbf{n})$, $\mathbf{n} \in S^{N-1}$ and wish to reconstruct $f(\mathbf{x})$. Figure 1 illustrates the geometry of the problem for CAT and NMR. $p(s, \mathbf{n})$ is defined by

$$\begin{aligned} p(s,\mathbf{n}) &= \int\limits_{s=\mathbf{x}\cdot\mathbf{n}} f(\mathbf{x})\,d\mathbf{x} \\ &= \int\limits_{R^N} f(\mathbf{x})\delta(s-\mathbf{x}\cdot\mathbf{n})\,d\mathbf{x} \end{aligned} \tag{2.2}$$

The Fourier Transform of $f(\mathbf{x})$ and its inverse are given by

$$F(\boldsymbol{\omega}) = \int\limits_{R^N} f(\mathbf{x})e^{-j\boldsymbol{\omega}\cdot\mathbf{x}}d\mathbf{x} \tag{2.3}$$

$$f(\mathbf{x}) = (2\pi)^{-N} \int\limits_{R^N} F(\boldsymbol{\omega})e^{j\boldsymbol{\omega}\cdot\mathbf{x}}d\boldsymbol{\omega} \tag{2.4}$$

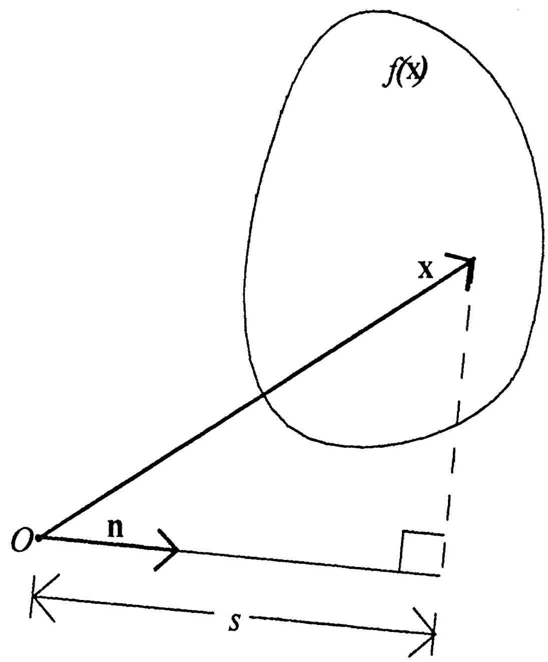
The problem of reconstruction from hyperplanar projections is most naturally handled in polar coordinates because the arguments to the projection $p(s, \mathbf{n})$ can be thought of as polar coordinates in projection space. In N-dimensional polar coordinates, the Fourier Transform of $f(r, \mathbf{m})$ and its inverse are given by

$$F(\rho,\boldsymbol{\alpha}) = 2^{-1} \int\limits_{S^{N-1}} \int\limits_{R} f(r,\mathbf{m})e^{-jr\rho\boldsymbol{\alpha}\cdot\mathbf{m}}|r|^{N-1}dr\,d\mathbf{m} \tag{2.5}$$

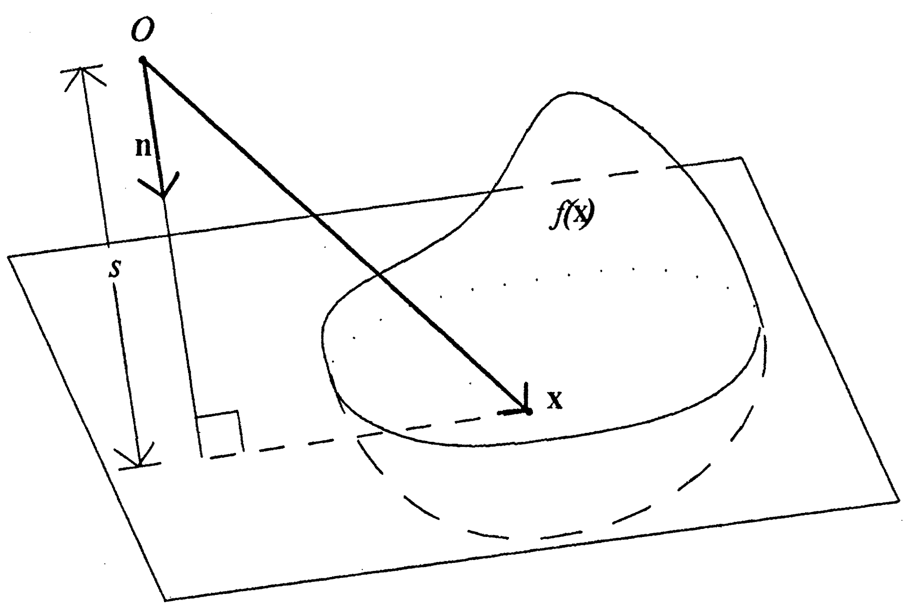
$$f(r,\mathbf{m}) = (2\pi)^{-N}2^{-1} \int\limits_{S^{N-1}} \int\limits_{R} F(\rho,\boldsymbol{\alpha})e^{jr\rho\boldsymbol{\alpha}\cdot\mathbf{m}}|\rho|^{N-1}d\rho\,d\boldsymbol{\alpha} \tag{2.6}$$

The N-dimensional Fourier Transform of $f(r, \mathbf{m})$ and the one-dimensional Fourier Transform of the projection $p(s, \mathbf{n})$ are related by the projection theorem (Appendix A).

$$F(\rho,\boldsymbol{\alpha}) = P(\rho,\boldsymbol{\alpha}) \tag{2.7}$$



(a) Projection geometry for CAT scans



(b) Projection geometry for NMR scans

Figure 1

So, if we examine the inverse Fourier Transform of $f(\mathbf{x})$ in polar coordinates, we get

$$f(\mathbf{x}) = (2\pi)^{-N} 2^{-1} \int_{S^{N-1}} \int_{R} F(\rho, \boldsymbol{\alpha}) e^{j\rho\boldsymbol{\alpha}\cdot\mathbf{x}} |\rho|^{N-1} d\rho \, d\boldsymbol{\alpha}$$

$$= \int_{S^{N-1}} \left[(2\pi)^{-N} 2^{-1} \int_{R} P(\rho, \boldsymbol{\alpha}) e^{j\rho\boldsymbol{\alpha}\cdot\mathbf{x}} |\rho|^{N-1} d\rho \right] d\boldsymbol{\alpha} \tag{2.8}$$

We see from (2.8) that $f(\mathbf{x})$ can be obtained by computing the expression in brackets, and smearing this function over all directions. This smearing is simply the backprojection-summation operation. Most authors (e.g. Chiu et.al. 1980, Lewitt 1983, Shepp 1980) normalize the backprojection operation by dividing by the surface area of a unit sphere in N dimensions. Let A_N be this factor.

$$A_N = \int_{S^{N-1}} d\boldsymbol{\alpha} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \tag{2.9}$$

With the inclusion of this normalization term, we can rewrite $f(\mathbf{x})$ as

$$f(\mathbf{x}) = \frac{1}{A_N} \int_{S^{N-1}} \left[A_N (2\pi)^{-N} 2^{-1} \int_{R} P(\rho, \boldsymbol{\alpha}) e^{j\rho\boldsymbol{\alpha}\cdot\mathbf{x}} |\rho|^{N-1} d\rho \right] d\boldsymbol{\alpha} \tag{2.10}$$

The bracketed expression is the inverse FT of a product. It can also be expressed as a convolution.

$$A_N (2\pi)^{-N} 2^{-1} \int_{R} P(\rho, \boldsymbol{\alpha}) e^{j\rho\boldsymbol{\alpha}\cdot\mathbf{x}} |\rho|^{N-1} d\rho$$

$$= (2\pi)^{-1} \int_{R} [\Gamma(\tfrac{N}{2})]^{-1} 2\pi^{\frac{N}{2}} P(\rho, \boldsymbol{\alpha}) (2\pi)^{1-N} 2^{-1} |\rho|^{N-1} e^{j\rho\boldsymbol{\alpha}\cdot\mathbf{x}} d\rho$$

$$= p(\boldsymbol{\alpha}\cdot\mathbf{x}, \boldsymbol{\alpha}) \otimes FT^{-1}\left\{ [\Gamma(\tfrac{N}{2})]^{-1} 2^{1-N} \pi^{1-\frac{N}{2}} |\rho|^{N-1} \right\} \qquad \text{where } \otimes = \text{convolution}$$

$$= p(s, \boldsymbol{\alpha}) \otimes g(s) \tag{2.11}$$

$$\text{Where } g(s) = FT^{-1}\left\{ [\Gamma(\tfrac{N}{2})]^{-1} 2^{1-N} \pi^{1-\frac{N}{2}} |\rho|^{N-1} \right\} \tag{2.12}$$

$$\text{with Fourier Transform } G(\rho) = [\Gamma(\tfrac{N}{2})]^{-1} 2^{1-N} \pi^{1-\frac{N}{2}} |\rho|^{N-1} \tag{2.13}$$

We will treat the cases of N even and odd separately.

Case N Even

The fact that N is even can be used to advantage by introducing the cosine transform.

$$
\begin{aligned}
g(s) &= (2\pi)^{-1}\int_{-\infty}^{\infty}[\Gamma(\tfrac{N}{2})]^{-1}2^{1-N}\pi^{1-\frac{N}{2}}|\rho|^{N-1}e^{js\rho}d\rho \\
&= [\Gamma(\tfrac{N}{2})]^{-1}2^{-N}\pi^{-\frac{N}{2}}\int_{0}^{\infty}\rho^{N-1}(e^{js\rho}+e^{-js\rho})d\rho \\
&= [\Gamma(\tfrac{N}{2})]^{-1}2^{1-N}\pi^{-\frac{N}{2}}\int_{0}^{\infty}\rho^{N-1}\cos(s\rho)d\rho \\
&= [\Gamma(\tfrac{N}{2})]^{-1}2^{1-N}\pi^{-\frac{N}{2}}\Gamma(N)\cos\left(\tfrac{N\pi}{2}\right)s^{-N} \\
&= 2^{1-N}\pi^{-\frac{N}{2}}\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}(-1)^{\frac{N}{2}}s^{-N}
\end{aligned}
\tag{2.14}
$$

Strictly speaking, the integral only converges for $0 < N < 1$ (Gradshteyn and Ryzhik 1980). It has been argued using convergence functions (Horn 1973) that the same form holds for all N. This is not quite correct.

This can be seen by considering the total area of $g(s)$, $\int_{-\infty}^{\infty} g(s)\,ds$. This must equal 0 since $G(\rho = 0) = 0$ for N even and positive. But, when N is even, $1/s^N$ is always positive and g(s) cannot possibly integrate to 0.

This discrepancy can be resolved by taking $g(s)$ to be a generalized function (Gel'fand and Shilov 1964, Lighthill 1958) rather than an ordinary function. Thus, $g(s)$ can be defined as

$$g(s) = \lim_{\epsilon\to 0} g_\epsilon(s)$$

$$
\text{Where } g_\epsilon(s) = \begin{cases} 2^{1-N}\pi^{-\frac{N}{2}}\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}(-1)^{\frac{N}{2}}\frac{1}{s^N} & \text{if } |s| > \epsilon \\ 2^{1-N}\pi^{-\frac{N}{2}}\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}(-1)^{\frac{N}{2}}\frac{1}{(1-N)\epsilon^N} & \text{if } |s| \le \epsilon \end{cases}
\tag{2.15}
$$

To simplify notation, we will write s^{-N} to denote the generalized function

$$
\lim_{\epsilon\to 0}\begin{cases} \frac{1}{s^N} & \text{if } |s| > \epsilon \\ \frac{1}{(1-N)\epsilon^N} & \text{if } |s| \le \epsilon \end{cases}
\tag{2.16}
$$

We will continue to write $\frac{1}{s^N}$ for the ordinary function.

The deblurring function blows up rapidly at the origin, for all values of N. This may introduce numerical problems when trying to actually compute the convolution.

Notice that $g(s)$ has an infinite region of support. Therefore the reconstruction of $f(\mathbf{x})$ at a single point will depend on all values of the projection $p(s, \mathbf{n})$. Reconstruction in this case is not local.

The above formula is only valid for N even. When N is odd, formally applying the formula yields 0 everywhere, with the possible exception of the origin, because the factor of $\cos(\frac{N\pi}{2})$ is 0. At the origin, we get 0 times a function that goes up as s^{-N}. Therefore the case of N odd must be treated separately to get meaningful results.[1]

Case N Odd

When N is odd we have from (2.13)

$$G(\rho) = [\Gamma(\tfrac{N}{2})]^{-1} 2^{1-N} \pi^{1-\frac{N}{2}} \rho^{N-1} \tag{2.17}$$

The differentiation operator has transform $j\rho$, and the $N-1$ derivative, N odd, has transform $\rho^{N-1}(-1)^{\frac{N-1}{2}}$. So,

$$g(s) = \frac{1}{\Gamma(\frac{N}{2})} 2^{1-N} \pi^{1-\frac{N}{2}} (-1)^{\frac{N-1}{2}} \frac{d^{N-1}}{ds^{N-1}} \tag{2.18}$$

In this case $g(s)$ is a differential operator. Its region of support is limited. Therefore the reconstruction of $f(\mathbf{x})$ at a single point will depend only on values of the projection $p(s, \mathbf{n})$ that include the point of interest. Reconstruction in this case is local.

Summary

In the convolution-backprojection method, each projection must be filtered before backprojection and summation. The filtering function is

$$g(s) = \begin{cases} 2^{1-N} \pi^{-\frac{N}{2}} \frac{\Gamma(N)}{\Gamma(\frac{N}{2})} (-1)^{\frac{N}{2}} s^{-N} & \text{if } N \text{ even} \\ 2^{1-N} \pi^{1-\frac{N}{2}} \frac{1}{\Gamma(\frac{N}{2})} (-1)^{\frac{N-1}{2}} \frac{d^{N-1}}{ds^{N-1}} & \text{if } N \text{ odd} \end{cases} \tag{2.19}$$

In CAT, $N = 2$

$$g(s) = -\frac{1}{2\pi} s^{-2} \tag{2.20}$$

In NMR, $N = 3$

$$g(s) = -\frac{1}{2\pi} \frac{d^2}{ds^2} \tag{2.21}$$

These special cases agree with (Chiu et.al. 1980), (Shepp 1980), and (Tanakaffi1979).

[1]Actually, if we consider $g(s)$ to be a function of N, it happens that it is an entire function, and therefore can be defined even when N is odd. Such an approach is taken in (Gel'fand and Shilov 1964) to obtain a differential operator when N is odd.

3. Backprojection and Summation Followed by Convolution

In this method, we first form the backprojection of the projection data, sum over all directions, and then filter. As before, the backprojection is normalized by A_N. The backprojection-summation $b(\mathbf{x})$ is obtained by

$$\begin{aligned} b(\mathbf{x}) &= A_N^{-1} \int_{S^{N-1}} p(\mathbf{x}\cdot\mathbf{n}, \mathbf{n})\, d\mathbf{n} \\ &= A_N^{-1} \int_{S^{N-1}} \int_{R^N} f(\mathbf{x}')\delta(\mathbf{x}\cdot\mathbf{n} - \mathbf{x}'\cdot\mathbf{n})\, d\mathbf{x}' d\mathbf{n} \\ &= \int_{R^N} f(\mathbf{x}') A_N^{-1} \int_{S^{N-1}} \delta(\mathbf{x}\cdot\mathbf{n} - \mathbf{x}'\cdot\mathbf{n})\, d\mathbf{n}\, d\mathbf{x}' \\ &= f(\mathbf{x}) \otimes h(\mathbf{x}) \end{aligned} \tag{3.1}$$

The backprojection-summation operation is equivalent to convolving the density function $f(\mathbf{x})$ with the blurring function $h(\mathbf{x})$. We need to find what $h(\mathbf{x})$ is. In Appendix D it is shown that

$$\int_{S^{N-1}} \delta(\mathbf{x}\cdot\mathbf{n})\, d\mathbf{n} = \frac{2\pi^{\frac{N-1}{2}}}{|\mathbf{x}|\Gamma(\frac{N-1}{2})} \tag{3.2}$$

Therefore the blurring function is

$$\begin{aligned} h(\mathbf{x}) &= \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{2\pi^{\frac{N-1}{2}}}{|\mathbf{x}|\Gamma(\frac{N-1}{2})} \\ &= \frac{1}{|\mathbf{x}|} \frac{\Gamma(\frac{N}{2})}{\sqrt{\pi}\Gamma(\frac{N-1}{2})} \end{aligned} \tag{3.3}$$

We would like to find another function, $g(\mathbf{x})$ that will undo the blurring caused by $h(\mathbf{x})$. g must be the convolutional inverse of h i.e. $g(\mathbf{x}) \otimes h(\mathbf{x}) = \delta(\mathbf{x})$. We observe that since h is symmetric, g must also be spherically symmetric. Therefore, in this section we may take $g = g(r)$. The best way to find g is to take Fourier Transforms. From above, we must have

$$G(\rho) = \frac{1}{H(\rho)} \tag{3.4}$$

Using the result in Appendix C concerning the FT of $|r|^k$, we have, with $k = -1$

$$\begin{aligned} H(\rho) &= \pi^{-\frac{1}{2}} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})} 2^{N-1} \pi^{\frac{N}{2}} \frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{1}{2})} |\rho|^{1-N} \\ &= 2^{N-1} \pi^{\frac{N}{2}-1} \Gamma(\tfrac{N}{2}) |\rho|^{1-N} \end{aligned} \tag{3.5}$$

$$G(\rho) = 2^{1-N} \pi^{1-\frac{N}{2}} \frac{1}{\Gamma(\frac{N}{2})} |\rho|^{N-1} \tag{3.6}$$

As before, we will treat the cases of N even and odd separately.

Case N Even

In the case of N even, we can use the results of Appendix C to determine the inverse FT of $G(\rho)$.

$$\begin{aligned} g(r) &= 2^{1-N}\pi^{1-\frac{N}{2}}\frac{1}{\Gamma(\frac{N}{2})}2^{N-1}\pi^{-\frac{N}{2}}\frac{\Gamma(\frac{N+N-1}{2})}{\Gamma(\frac{-N+1}{2})}r^{-N-N+1} \\ &= \pi^{1-N}\frac{\Gamma(N-\frac{1}{2})}{\Gamma(\frac{N}{2})\Gamma(\frac{1-N}{2})}r^{1-2N} \end{aligned} \tag{3.7}$$

Most of the comments that were made about convolution-backprojection for N even also apply here. Specifically, the deblurring function blows up at the origin. Also, its region of support is infinite rather than local. As before, $g(r)$ is a generalized function.

The above formula is only valid for N even. When N is odd, the factor $\Gamma(\frac{1-N}{2})$ is undefined. Therefore, the case of N odd must be treated separately.

Case N Odd

This is solved in exactly the same manner as in the convolution-backprojection method. When N is odd we have from (3.6)

$$G(\rho) = 2^{1-N}\pi^{1-\frac{N}{2}}\frac{1}{\Gamma(\frac{N}{2})}\rho^{N-1} \tag{3.8}$$

The Laplacian operator, ∇^2, has transform $-\rho^2$. So,

$$g(r) = 2^{1-N}\pi^{1-\frac{N}{2}}\frac{1}{\Gamma(\frac{N}{2})}(-1)^{\frac{N-1}{2}}\nabla^{N-1} \tag{3.9}$$

Where ∇^{N-1} indicates to take the Laplacian $\frac{N-1}{2}$ times. This is equivalent to

$$\nabla^{2k} = \sum_{i_1}\sum_{i_2}\cdots\sum_{i_k}\frac{\partial^{2k}}{\partial x_{i_1}^2\partial x_{i_2}^2\cdots\partial x_{i_k}^2} \tag{3.10}$$

This is the same form as was obtained with the convolution-backprojection method, except the differentiation in one dimension has been replaced by a series of Laplacians. As before, $g(r)$ is a differential operator. Its region of support is limited. Therefore the reconstruction of $f(\mathrm{x})$ at a single point will depend only on values of the backprojection $b(\mathrm{x})$ in a small neighborhood about the point, which in turn depend depend only on values of the projection $p(s,\mathrm{n})$ that include the point of interest. Reconstruction in this case is local.

Summary

In the backprojection-convolution method, each projection must be filtered after backprojection and summation. The filtering function is

$$g(r) = \begin{cases} \pi^{1-N} \frac{\Gamma(N-\frac{1}{2})}{\Gamma(\frac{N}{2})\Gamma(\frac{1-N}{2})} r^{1-2N} & \text{if } N \text{ even} \\ 2^{1-N} \pi^{1-\frac{N}{2}} \frac{1}{\Gamma(\frac{N}{2})} (-1)^{\frac{N-1}{2}} \nabla^{N-1} & \text{if } N \text{ odd} \end{cases} \tag{3.11}$$

In CAT, $N = 2$

$$g(r) = -\frac{1}{4\pi} r^{-3} \tag{3.12}$$

In NMR, $N = 3$

$$g(\mathbf{x}) = -\frac{1}{2\pi} \nabla^2 \tag{3.13}$$

These special cases also agree with (Chiu et.al. 1980), (Shepp 1980), and (Tanaka 1979).

4. Comparison of Even and Odd Dimensional Reconstruction

In this chapter we will discuss why the reconstruction equations look so different in even and odd dimension. To make the discussion concrete, consider the convolution-backprojection method. To repeat, we have (2.13)

$$G(\rho) = 2^{1-N}\pi^{1-\frac{N}{2}}\frac{1}{\Gamma(\frac{N}{2})}|\rho|^{N-1}$$

The filtering function is (2.19)

$$g(s) = \begin{cases} 2^{1-N}\pi^{-\frac{N}{2}}\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}(-1)^{\frac{N}{2}}s^{-N} & \text{if } N \text{ even} \\ 2^{1-N}\pi^{1-\frac{N}{2}}\frac{1}{\Gamma(\frac{N}{2})}(-1)^{\frac{N-1}{2}}\frac{d^{N-1}}{ds^{N-1}} & \text{if } N \text{ odd} \end{cases}$$

It is possible to explain the differences by noting that $|\rho|^{N-1} = \rho^{N-1}$ when N is odd. This leads to the second form above, in which the filtering operation is equivalent to differentiation. When N is even, then $|\rho|^{N-1} = \rho^{N-1}\operatorname{sgn}(\rho)$. Since multiplication in the transform domain corresponds to convolution in the spatial domain, we can try to perform the convolution directly. So let us redo the analysis for N even, using this approach.

For N even, we have

$$G(\rho) = 2^{1-N}\pi^{1-\frac{N}{2}}\frac{1}{\Gamma(\frac{N}{2})}\rho^{N-1}\operatorname{sgn}(\rho) \tag{4.1}$$

Now the inverse transform of ρ^{N-1} is

$$FT^{-1}\{\rho^{N-1}\} = (-j)^{N-1}\frac{d^{N-1}}{ds^{N-1}} \tag{4.2}$$

And the inverse transform of $\operatorname{sgn}(\rho)$ is

$$FT^{-1}\{\operatorname{sgn}(\rho)\} = \frac{j}{\pi s} \tag{4.3}$$

Putting it all together, we get

$$\begin{aligned} g(s) &= 2^{1-N}\pi^{1-\frac{N}{2}}\frac{1}{\Gamma(\frac{N}{2})}(-j)^{N-1}\frac{d^{N-1}}{ds^{N-1}}\left[\frac{j}{\pi s}\right] \\ &= 2^{1-N}\pi^{-\frac{N}{2}}\frac{1}{\Gamma(\frac{N}{2})}(-1)^{N-1}j^{N}(-1)^{N-1}(N-1)!\,s^{-N} \\ &= 2^{1-N}\pi^{-\frac{N}{2}}\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}(-1)^{\frac{N}{2}}s^{-N} \end{aligned} \tag{4.4}$$

This is exactly the result obtained through direct transformation using the formula in Appendix C. This shows that the basic difference between even and odd dimensions is that raising the magnitude of ρ to an odd power is equivalent to differentiating the function $\frac{1}{s}$ in the spatial domain. To make the similarity more apparent, we can rewrite the correction function as

$$g(s) = \begin{cases} 2^{1-N}\pi^{1-\frac{N}{2}} \frac{1}{\Gamma(\frac{N}{2})} j^{N-1} \frac{d^{N-1}}{ds^{N-1}} \left(\frac{j}{\pi s}\right) & \text{if } N \text{ even} \\ 2^{1-N}\pi^{1-\frac{N}{2}} \frac{1}{\Gamma(\frac{N}{2})} j^{N-1} \frac{d^{N-1}}{ds^{N-1}} \delta(s) & \text{if } N \text{ odd} \end{cases} \tag{4.5}$$

The above expressions for $g(s)$ form a Hilbert Transform (Bracewell 1965) pair. The Hilbert Transform of an even function is an odd function, and the transform of an odd function is even. Thus the Hilbert Transform can be regarded as correcting the antisymmetric differentiation when N is even.

This analysis does not carry over to the backprojection-convolution case. This is because it is not possible to define $\text{sgn}(\rho)$ in more than one dimension. Nor can the Hilbert Transform be defined. However, the underlying principle is the same. That is, the correction function must be spherically symmetric in all cases. When the dimensionality of the problem is even, this leads naturally to derivatives of even order. When the dimensionality of the problem is odd, we are not allowed to use the odd derivatives, because they are not symmetric functions.

Acknowledgment: I are grateful to B.K.P. Horn for suggesting this field of research, and for his many valuable suggestions and discussions. Thanks also to A. Yuille for his comments and suggestions.

Appendix A

In Appendix A we prove the projection theorem (Mersereau and Oppenheim 1974) in polar coordinates. Let $p(s, \boldsymbol{\alpha})$ be the projection of $f(\mathbf{x})$ along a plane perpendicular to $\boldsymbol{\alpha}$. Then the one-dimensional Fourier Transform of p and the N-dimensional Transform of f are related as follows.

$$\begin{aligned}
P(\rho, \boldsymbol{\alpha}) &= \int_{R} p(s, \boldsymbol{\alpha}) e^{-js\rho} ds \\
&= \int_{R} \int_{R^N} f(\mathbf{x}) \delta(s - \mathbf{x} \cdot \boldsymbol{\alpha}) \, d\mathbf{x} e^{-js\rho} ds \\
&= \int_{R} 2^{-1} \int_{S^{N-1}} \int_{R^N} f(r, \mathbf{m}) \delta(s - r\mathbf{m} \cdot \boldsymbol{\alpha}) |r|^{N-1} dr \, d\mathbf{m} e^{-js\rho} ds \\
&= 2^{-1} \int_{S^{N-1}} \int_{R} f(r, \mathbf{m}) |r|^{N-1} e^{-jr\rho \mathbf{m} \cdot \boldsymbol{\alpha}} dr \, d\mathbf{m} \\
&= F(\rho, \boldsymbol{\alpha})
\end{aligned} \tag{A.1}$$

The FT of $p(s, \boldsymbol{\alpha})$ equals the FT of $f(\mathbf{x})$ evaluated in the $\boldsymbol{\alpha}$ direction.

Appendix B

In Appendix B we derive the Fourier Transform of a spherically symmetric function and the inverse transform. Let $f(\mathbf{x}) = f(r)$ with Fourier Transform $F(\boldsymbol{\omega}) = F(\rho)$. In polar coordinates the elements of $\mathbf{x}$ and $\boldsymbol{\omega}$ can be defined by

$$\begin{aligned} x_0 &= r \sin\theta_0 \cdots \sin\theta_{N-2} \\ x_k &= r\cos\theta_{k-1} \prod_{i=k}^{N-2} \sin\theta_i, \quad 1 \le k \le N-2 \\ x_{N-1} &= r\cos\theta_{N-2} \end{aligned}$$

$$\begin{aligned} \omega_0 &= \rho \sin\alpha_0 \cdots \sin\alpha_{N-2} \\ \omega_k &= \rho\cos\alpha_{k-1} \prod_{i=k}^{N-2} \sin\alpha_i, \quad 1 \le k \le N-2 \\ \omega_{N-1} &= \rho\cos\alpha_{N-2} \end{aligned}$$

The Fourier Transform is given by

$$F(\boldsymbol{\omega}) = \int_{R^N} f(\mathbf{x}) e^{-j\mathbf{x}\cdot\boldsymbol{\omega}} d\mathbf{x} \tag{B.1}$$

$$F(\rho) = \int_{-\infty}^{\infty} \int_0^{\pi} \cdots \int_o^{\pi} f(r) e^{-jr\rho \left(\begin{array}{l} \sin\theta_0 \cdots \sin\theta_{N-2} \sin\alpha_0 \cdots \sin\alpha_{N-2} + \\ \cos\theta_0 \sin\theta_1 \cdots \sin\theta_{N-2} \cos\alpha_0 \sin\alpha_1 \cdots \sin\alpha_{N-2} \\ + \cdots + \\ \cos\theta_{N-2}\cos\alpha_{N-2} \end{array} \right)} r^{N-1} \sin^1\theta_1 \cdots \sin^{N-2}\theta_{N-2} d\theta_0 \cdots d\theta_{N-2}\, dr \tag{B.2}$$

The transform of a spherically symmetric function is also spherically symmetric. So we can take $\alpha_i = 0$ without loss of generality. The integration over r need only be performed from 0 on up, as long as a factor of 2 is included. Using the facts that

$$\int_0^{\pi} \sin^k\theta\, d\theta = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{i+1}{2})}{\Gamma(\frac{i}{2}+1)}$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

and

$$\int_0^{\pi} e^{j\beta\cos\theta} \sin^{2v}\theta\, d\theta = \sqrt{\pi}\left(\frac{2}{\beta}\right)^v \Gamma(v+\tfrac{1}{2}) J_v(\beta)$$

We have

$$
\begin{aligned}
F(\rho) &= 2\int_0^\infty f(r)r^{N-1}dr \int_0^\pi \sin^0\theta_0\, d\theta_0 \cdots \int_0^\pi \sin^{N-3}\theta_{N-3}\, d\theta_{N-3} \int_0^\pi e^{-jr\rho\cos\theta_{N-2}} \sin^{N-2}\theta_{N-2}\, d\theta_{N-2} \\
&= 2\int_0^\infty f(r)r^{N-1}dr \prod_{i=0}^{N-3}\left(\pi^{\frac{1}{2}}\frac{\Gamma(\frac{i+1}{2})}{\Gamma(\frac{i}{2}+1)}\right)\pi^{\frac{1}{2}}\left(\frac{2}{r\rho}\right)^{\frac{N}{2}-1}\Gamma(\tfrac{N-1}{2})J_{\frac{N}{2}-1}(r\rho) \\
&= (2\pi)^{\frac{N}{2}}\rho^{1-\frac{N}{2}}\int_0^\infty r^{\frac{N}{2}}J_{\frac{N}{2}-1}(r\rho)f(r)\,dr \qquad \text{(B.3)}
\end{aligned}
$$

Interchanging r and ρ and dividing by $(2\pi)^N$ gives us the inverse Fourier Transform

$$
f(r) = (2\pi)^{-\frac{N}{2}}r^{1-\frac{N}{2}}\int_0^\infty \rho^{\frac{N}{2}}J_{\frac{N}{2}-1}(r\rho)F(\rho)\,d\rho \qquad \text{(B.4)}
$$

Appendix C

In Appendix C we apply the spherically symmetric form of the Fourier Transform to the function $f(r) = r^k$. Since $r = |\mathbf{x}|$ and $\rho = |\omega|$, we will assume in this appendix that r and ρ are non-negative. From Appendix B we have

$$\begin{aligned} F(\rho) &= (2\pi)^{\frac{N}{2}} \rho^{1-\frac{N}{2}} \int_0^\infty r^{\frac{N}{2}} J_{\frac{N}{2}-1}(r\rho) f(r)\, dr \\ &= (2\pi)^{\frac{N}{2}} \rho^{1-\frac{N}{2}} \int_0^\infty r^{\frac{N}{2}} J_{\frac{N}{2}-1}(r\rho) r^k dr \\ \text{Let } z = r\rho \qquad & \\ &= (2\pi)^{\frac{N}{2}} \rho^{-k-N} \int_0^\infty z^{\frac{N}{2}+k} J_{\frac{N}{2}-1}(z)\, dz \\ &= 2^{N+k} \pi^{\frac{N}{2}} \frac{\Gamma(\frac{N+k}{2})}{\Gamma(-\frac{k}{2})} \rho^{-N-k} \end{aligned} \tag{C.1}$$

Using the fact that

$$\int_0^\infty x^q J_p(x)\, dx = 2^q \frac{\Gamma(\frac{p+q+1}{2})}{\Gamma(\frac{p-q+1}{2})}$$

Interchanging r and ρ and dividing by $(2\pi)^N$ gives us the inverse transform of $F(\rho) = \rho^k$.

$$f(r) = 2^k \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+k}{2})}{\Gamma(-\frac{k}{2})} r^{-N-k} \tag{C.2}$$

Appendix D

In Appendix D we derive an expression for $\int_{S^{N-1}} \delta(\mathbf{x}\cdot\mathbf{n})\,d\mathbf{n}$ This integral is obviously independent of the direction of $\mathbf{x}$, but is dependent on its magnitude. We can express $\mathbf{n}$ in polar coordinates, and let $\mathbf{x}$ lie along the last component of $\mathbf{n}$, since the orientation of $\mathbf{n}$ is arbitrary. Define $\mathbf{n}$ by

$$\begin{aligned} n_0 &= \sin\theta_0 \cdots \sin\theta_{N-2} \\ n_k &= \cos\theta_{k-1} \prod_{i=k}^{N-2} \sin\theta_i, \quad 1 \le k \le N-2 \\ n_{N-1} &= \cos\theta_{N-2} \end{aligned}$$

Using the facts that

$$\int \delta(kx)\,dx = \frac{1}{k}$$

$$\int_0^{\pi} \sin^k\theta\,d\theta = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{i+1}{2})}{\Gamma(\frac{i}{2}+1)}$$

and

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

We have

$$\begin{aligned} \int_{S^{N-1}} \delta(\mathbf{x}\cdot\mathbf{n})\,d\mathbf{n} &= \int_0^{2\pi}\int_0^{\pi}\cdots\int_o^{\pi} \delta(|\mathbf{x}|\cos\theta_{N-2})\sin^1\theta_1\cdots\sin^{N-2}\theta_{N-2}d\theta_0\cdots d\theta_{N-2} \\ &= 2|\mathbf{x}|^{-1}\prod_{i=0}^{N-3}\left(\pi^{\frac{1}{2}}\frac{\Gamma(\frac{i+1}{2})}{\Gamma(\frac{i}{2}+1)}\right) \\ &= \frac{2\pi^{\frac{N-1}{2}}}{|\mathbf{x}|\Gamma(\frac{N-1}{2})} \end{aligned} \tag{D.1}$$

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