MATHEMATICAL ANALYSIS SECTION

MAS REPORT NO. 1

"THE TRW-130 (AN/UYK-1) APPLIED TO TRACKING RADAR DATA PROCESSING"

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THE TRW-130 (AN/UYK-1) APPLIED TO TRACKING RADAR DATA PROCESSING

1. INTRODUCTION

A program presently exists to operate AN/UYK-1 computers which assist radars in the Atlantic Missile Range in acquisition and tracking of a target. This program is described along with mathematical details necessary for the implementation of such a program. Also, possible areas of sophistication are indicated to improve the effectiveness of such a program in some respects. These modifications, however, require increased computing time in general.

1.1 Problem Statement

The basic problem is to automate those functions necessary for a tracking radar to acquire a target and then begin automatic tracking of this target. It is desirable for this target acquisition to occur as soon as possible in order that the radar will be able to output good position data which may then be used by a large central computer in such areas as the range safety functions (i.e., impact prediction). The application of a digital computer to this problem allows a high speed (real time), accurate solution.

Several features of the AN/UYK-1 make it highly suited to this problem. First, it is a relatively high speed computer which, along with the adaptability inherent in the stored-logic feature, allows the necessary functions to be performed in the short cycle time (100 ms) allowed. Secondly, it is "ruggedized" which, due to the relative remoteness of the sites, eases the associated transportation and environmental problems. Thirdly, the AN/UYK-1 has a sophisticated interrupt system which allows response to the associated external devices in synchronism with real time with a minimum of programming problems.

2. GENERAL DESCRIPTION OF THE ATLANTIC MISSILE RANGE ACQUISITION SYSTEM (ARCAS)

The basic make-up of the system is:

1. Six Radar-Buffer-AN/UYK-1 Computer sites on the Atlantic Missile Range.

2. A submarine cable for data transmission from site-to-site.

3. A high-speed data transmission system from radars to a larger computer at the Cape Canaveral site. This computer will transmit data over the submarine cable just as an AN/UYK-1.

The general objective is to be able to have a site begin tracking a missile, obtain position data from this radar site, and then transmit this data downrange. This data must be presented to a downrange radar such that it will be properly directed at the missile in order to allow it to lock-on the target and go into an automatic tracking mode.

Also, the position data obtained from the various site radars will be sent directly to the central computer which will use the data for impact prediction and associated range safety functions in addition to transmitting position data as an AN/UYK-1 does. Also, the central computer and each AN/UYK-1 must present data suitable for plotting. Namely, in cartesian (X, Y, Z) coordinates referenced to the local radar site.

The AN/UYK-1 computer fulfills these objectives by performing the following functions:

1. Read range, azimuth, and elevation data (RAE) from a buffer which is connected between the local radar and the computer.
2. Test this data for reasonableness.
3. Apply certain corrections to the data.
4. Convert it to a cartesian form referenced to the local site.
5. Update this data for various time delays in the system.
6. Determine which one of five coordinate system origins is approximately closest to the target.
7. Transform the data to this common reference origin.
8. Insert a set of ID bits to specify this origin.
9. Scale the data to appropriate units and round to 12 bits.
10. Output the data along with a scale code and origin code to a buffer which will transmit it to the submarine cable.

Items (1) through (10) are called the designation or tracking mode.

11. Read cartesian (X, Y, Z) data from a buffer which is referenced to some common origin.
12. Scale the data based on scaling code bits.
13. Determine this origin from the ID bits.
14. Select the proper matrix to transform this data to the local site.
15. Perform a reasonableness check on the data.
16. Output this local X, Y, Z data to a buffer for plotting purposes.
17. Convert this cartesian data to polar form (RAE).
18. Update this data for time delays.
19. Determine a scan mode recommendation.
20. Apply corrections to the data.
21. Output this R, A, E data along with scan recommendation bits to a buffer which will give the data to the radar.

Items (11) through (21) are known as the acquisition mode.

The requirements that must be satisfied while performing these functions are:

1. The entire program, including both modes, must be able to be executed in less than 100 milliseconds.
2. The X, Y, Z data from the designation mode must be output to the buffer prior to the 50 millisecond point.
3. The entire program must be contained in 5460 cells or less.
4. A provision for an alternate mode called the fixed mode must exist which would result in transforming data to an origin referenced at the local site rather than a common range origin.

3. DETAILED ARCAS PROGRAM DESCRIPTION

3.1 General

Presented in this section is the basic program description with many mathematical details omitted. Reference may be made to Section 4 as needed by the reader to clarify points not precisely described in the following discussion of the program.

Every 100 milliseconds, cartesian coordinates referenced to the common coordinate system with a code indicating which station is in the best position for tracking the object at a given time are output to the other five units. From this data with the associated identification (ID) code, one station can begin tracking, and output to the radar, polar coordinates in the local coordinate system with a scan mode recommendation. When radar data is read, it contains a code bit, designated CS, which determines whether the local transformation matrix is to be used. This bit is controlled via a switch on the radar operator's console.

Each station has an assigned ID and matrix elements which are designated as the local coordinate transformation for that installation. For example, Patrick Air Force Base might have an ID of 4. Then matrix 4 and vector 4 would be stored as the local matrix transformation at the Patrick installation, while Cape Canaveral might have an ID of 3; hence, matrix 3, vector 3 would be the local matrix transformation at the Canaveral Station.

An OT bit code, "on track," is input with channel B radar data. When the radar is on track, no scan mode recommendation is made. Otherwise, the program recommends a spiral or one of two raster scan modes. The rate of change of azimuth and elevation angles determine whether the spiral or raster scan mode is recommended. The spiral mode traverses equal azimuth and elevation angles and begins searching in the center and spirals outward. The raster scan mode, a slower process, begins searching at one corner of the pattern. The two raster scan modes differ only in the rate of change of azimuth. The method for scanning is only allowed to be changed every minute.

There are two starting points for the operational program. One sets up one set of matrix elements to rotate the X and Y coordinates to an earth centered vertical orientation for plotter output. The second entry sets up an identity transformation to out put X and Y oriented to the radar tangent plane.

The entire program is executed in a maximum of about 85 ms and requires about 4200 cells of storage. Operations are performed in either single or double precision as dictated by the precision of the input or output data. The input data is, in the case of the AN/FPS-16 radar, 20 bits, 19 bits, and, 19 bits for range, azimuth and elevation respectively. This requires double precision operations to be performed in most of the designation mode operations such as the polar to cartesian coordinate conversion. The final X, Y, Z output to be transmitted via the submarine cable, however, is 12 bits and a sign bit which allows certain scaling and output operations to be performed in single precision. The input to the acquisition mode is therefore 12 bits and a sign for X, Y, Z while the output for the radar is 15 bits, 13 bits, and 13 bits for R, A, and E respectively. This allows many functions of the acquisition mode, such as computation of A and E from X, Y, Z, to be performed in single precision. Some double precision arithmetic is required, however, and at times the necessary scaling of the numbers makes double precision operations convenient although not absolutely necessary.

3.2 Input-Output and Real Time Orientation

Channel A inputs, through the buffers, the X, Y, Z data and the ID and SC codes, all needed in the acquisition mode. Channel A output consists of the computed X, Y, Z (both for transmission and for plotting) and the ID and scale code (SC) determined in the designation mode.

Channel B inputs the R, A, E data and the CS and OT codes used in the designation mode. The output on channel B is R, A, E and the scan mode (SM) recommendation which is calculated in the acquisition mode.

All input and output is performed in response to interrupts from the buffering devices and full use is made of the interrupt capabilities of the AN/UYK-1. This interrupt system also allows a limited self checking operation to be performed on the system. That is, a check may be performed at the conclusion of a program cycle to determine if the correct number of interrupts were received from each channel. If not, the results of that cycle may be incorrect, and synchronism with real time may have been lost. However, instead of a collapse of the complete system, if this was merely an intermittent failure, the AN/UYK-1 may reorient itself in the real time cycle and correct operation is recovered with possible loss of only a few cycles of data.

This real time orientation is also the first step performed by the program when the system is first started. The orientation is accomplished by checking for the existence of input and output data requests (which remain in effect once initiated by the buffers), counting elapsed time to the next requests, and finally determining a channel A input data request which begins the program cycle.

3.3 Designation Mode

Data Input

Channel A data is input and the codes are extracted. Data is scaled by a power of 10, according to the scale code (SC) read with the data, and the units are changed for internal usage. Channel B data is input and the codes are removed. Data is stored, appropriately scaled, as specified by the program.

Validity Test and Extrapolation

A counter, which is incremented every cycle, indicates whether enough points have been input for validity checking and for extrapolation. These functions are bypassed until the eleventh cycle.

An average first difference is computed over the five most recent data points. This increment is then used for a single cycle extrapolation to use as a check value in the validity test, and also for time delay extrapolations.

If a piece of data fails the validity test, an error indicator is set, and the check value is employed in subsequent computations for that cycle. However, the next two data points are not checked and are accepted for use in computations. The third point is again checked.

Refraction and Bias Correction

The program next adds a refraction correction, computed during the previous cycle in the designation mode. Constant bias corrections are added to R, A, and E.

Polar to Cartesian Conversion

The sine and cosine of azimuth and elevation are computed and then used to compute X, Y, Z. The intermediate product ALPHA = R cos E is stored for the refraction correction calculations in the designation mode. X, Y, Z are stored with the same binary scaling as range.

Coordinate Transformation to Common

If the coordinate system bit (CS), which is read in with channel B data, is zero the local coordinate transformation is used. Otherwise, the reference origin counter is tested. If the counter is negative, then X-Y is compared against four barriers. The barriers represent imaginary planes dividing the tracking range of one station from the next one. When X-Y is less than bar I, Matrix I is used in the transformation, I is stored in identification bits (ID), and a program transfer is set so that the program automatically uses the matrix I transformation for the next 9 cycles. After completion of 10 cycles, X-Y is again compared against the barriers and a new I is found.

Maximum Coordinates

Following the coordinate system transformation, the program determines whether X, Y, or Z is the largest coordinate. The largest coordinate is compared against stored constants for finding a scale factor to reduce X, Y, Z to one word format. The scale code is stored for output.

Channel A Output

The scaled data has the ID and scale code added and is formatted and output on channel A.

3.4 Acquisition Mode

Coordinate Transformation to Local System

According to the ID input with the channel A data, a set of matrix and vector elements is selected for the transformation from the common to the local coordinate system. The data also has an additional transformation applied for proper orientation of X and Y for the plotter. These two values are then formatted and stored for output to the plotter at a later time.

Validity Check

A validity check is performed in the same manner as for the R, A, E data in the designation mode. A slight difference occurs since the expected range of random errors in the R, A, E data is known, but the random errors in the X, Y, Z are a function of the geometry of the situation and the allowed error for X, Y, Z is therefore based on the computed first difference. The procedure is explained in greater detail in Section 4.3. The time delay extrapolations must be delayed until R, A, and E are available since different delays are compensated for in R than in A and E.

Cartesian to Polar Conversion

Range is computed from the square root of the squares of X, Y, and Z. Azimuth is the arctangent of the quotient of X over Y. Elevation is the arcsine of Z divided by R. The quadrant for azimuth is determined and the two angles are converted from radians to units of fractions of a circle used for channel B input.

Refraction Correction Determination and Bias Corrections

In the refraction correction calculations, cotE is computed from R cos E/Z, where R cos E is available from the designation mode calculations. Next, a table look-up through thirteen values is performed with Z as an argument to find a factor $(N_s - \bar{N})(A)$ where A is a scale constant. See Section 4.4 for details. Finally, $\Delta E = (N_s - \bar{N}) \cot E$ which is then added to E. Constant bias corrections are next added to R, A, and E.

Extrapolation

R, A, and E first differences are now computed and used for extrapolation. A and E are extrapolated an additional cycle since the acquisition servo units for A and E work as a function of their error from the desired position.

Scan Mode Recommendations

Provided that on-track bit is not set and that time delay counter shows that one minute or more have elapsed since the last scan mode recommendation, a new scan mode is selected. The average differences for azimuth compared against the average differences for elevation and against a predetermined constant provides the method for making a raster scan mode or spiral scan mode recommendation. That is, the relative rates of change of elevation and azimuth are the selection criteria.

Data Output

Range units are converted for output. The two angles are changed to one word format and scan mode bits are added to the data before formatting for output to the buffer. The local plotter data is output followed by the acquisition data.

4. MATHEMATICAL DETAILS

4.1 Coordinate Conversions

Radar to Cartesian Coordinates (RAE⟶XYZ)

Assume azimuth (A) is measured from north, positive in a clockwise direction. It is assumed that if in some unusual situation a target passes directly overhead such that the elevation angle (E) exceeds 90 degrees, then the azimuth becomes 180 degrees out of phase. That is, the radar remains stationary in azimuth while the target passes directly overhead. In other words, there is a discontinuity (a singular point) at E = 90 degrees. The equations to convert from radar to cartesian coordinates then are:

$$\begin{aligned} x &= R \cos E \sin A \\ y &= R \cos E \cos A \\ z &= R \sin E \end{aligned} \qquad (4.1)$$

The change in the sign of cos E when E exceeds 90 degrees accounts for the effect of A becoming 180 degrees out of phase.

Cartesian to Radar Coordinates (XYZ⟶RAE)

The equations for this conversion are:

$$\begin{aligned} R &= \sqrt{x^2 + y^2 + z^2} \\ E &= \arcsin (z/R) \\ A' &= \arctan (y/x) \end{aligned} \qquad (4.2)$$

The proper quadrant for A is determined by a check of the signs of x and y:

x, y	
+, +	$A = 90^{\circ} - A'$
-, +	$A = 90^{\circ} + A'$
-, -	$A = 270^{\circ} - A'$
+, -	$A = 270^{\circ} + A'$

4.2 Affine (Site to Site) Transformation

Notation: Rotations about the various axes are as follows:

$$R_{\Theta x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta \\ 0 & \sin\Theta & \cos\Theta \end{pmatrix} \qquad R_{-\Theta x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & \sin\Theta \\ 0 & -\sin\Theta & \cos\Theta \end{pmatrix}$$

$$R_{\Theta y} = \begin{pmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{pmatrix} \qquad R_{\Theta z} = \begin{pmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The affine transformation from site i to site j is denoted by

$$(x)_j = A_{ij}(x)_i + T_{ij} \tag{4.3}$$

where

$$(x)_i = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_i$$

and A_{ij} is the rotation matrix with T_{ij} being the translation vector. A position vector with a prime indicates earth centered vertical orientation and is related to (x) by

$$(x)'_i = R_{-\delta_i x}(x)_i \tag{4.4}$$

Also define:

ϕ_i = latitude of site i

λ_i = longitude of site i

$\Delta\lambda = \lambda_i - \lambda_j$

s_i = distance to the equatorial plane along the normal to the tangent plane at site i.

d_i = distance to center of earth along equatorial plane from intersection of tangent plane normal with equatorial plane

h_i = height above sea level of site i

δ_i = angle between tangent plane normal and earth centered vertical axis at site i

See Figure 1 for an illustration of these quantities.

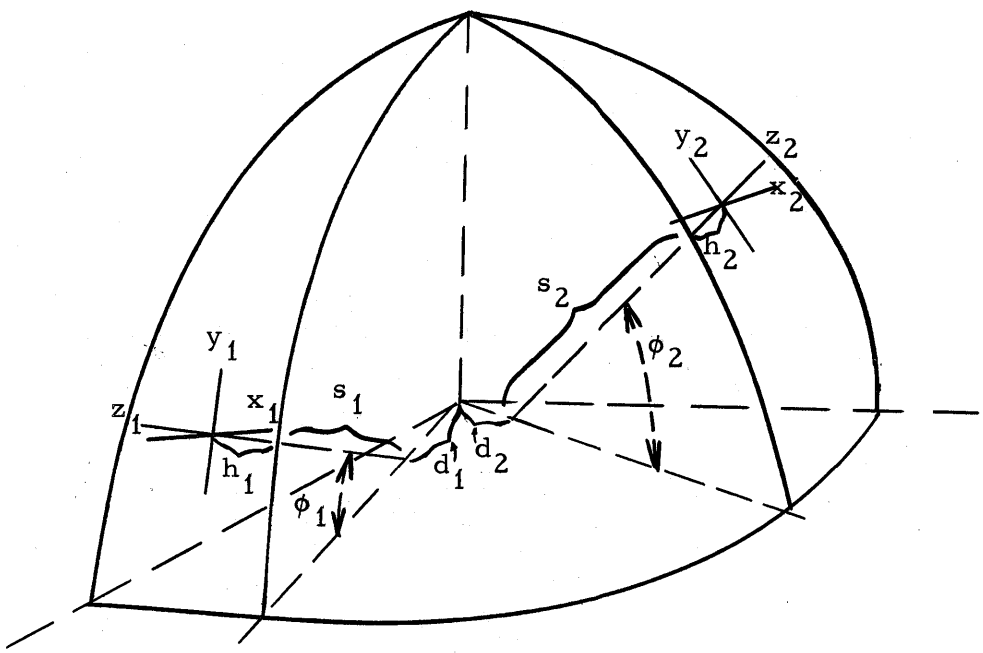

Figure 1. Site to Site Coordinate Transformation

Affine Transformation: The breakdown of the transformation from a tangent plane oriented system to an earth centered vertical system is

$$(x)_j' = R_{-\delta_i x}\left[R_{\phi_j x}\left[R_{\Delta\lambda y}\left[R_{-\phi_i x}\left[(x)_i + \begin{pmatrix} 0 \\ 0 \\ h_i + s_i \end{pmatrix}\right] + \begin{pmatrix} 0 \\ 0 \\ d_i \end{pmatrix}\right] - \begin{pmatrix} 0 \\ 0 \\ d_j \end{pmatrix}\right] - \begin{pmatrix} 0 \\ 0 \\ h_j + s_j \end{pmatrix}\right] \quad (4.5)$$

This is the transformation used in the designation mode in going from the radar oriented cartesian system to the earth centered vertical orientation in which data is transmitted via the submarine cable.

To get Eq. (4. 5) in the form of Eq. (4. 3), one combines terms and obtains

$$(x)_j' = R_{-\delta_j x} R_{\phi_j x} R_{\Delta\lambda y} R_{-\phi_i x} (x)_i + R_{-\delta_j x} R_{\phi_j x} R_{\Delta\lambda y} R_{-\phi_i x} \begin{pmatrix} 0 \\ 0 \\ h_i + s_i \end{pmatrix}$$

$$+ R_{-\delta_j x} R_{\phi_j x} R_{\Delta\lambda y} \begin{pmatrix} 0 \\ 0 \\ d_i \end{pmatrix} - R_{-\delta_j x} R_{\phi_j x} \begin{pmatrix} 0 \\ 0 \\ d_j \end{pmatrix} - R_{-\delta_j x} \begin{pmatrix} 0 \\ 0 \\ h_i + s_i \end{pmatrix} \qquad (4.6)$$

Then

$$A_{ij} = R_{\phi_j x} R_{\Delta\lambda y} R_{-\phi_i x} \qquad (4.7)$$

Letting $c\phi = \cos\phi$ and $s\phi = \sin\phi$, the rotation matrix is

$$A_{ij} = \begin{pmatrix} c\Delta\lambda & s\Delta\lambda s\phi_i & -s\Delta\lambda c\phi_i \\ -s\phi_j \, s\Delta\lambda & c\phi_j c\phi_i + s\phi_j c\Delta\lambda s\phi_i & c\phi_j s\phi_i - s\phi_j c\Delta\lambda c\phi_i \\ c\phi_j \, s\Delta\lambda & s\phi_j \, c\phi_i - c\phi_j c\Delta\lambda s\phi_i & s\phi_j s\phi_i + c\phi_j c\Delta\lambda c\phi_i \end{pmatrix} \qquad (4.8)$$

Let

$$B_{ij} = R_{\phi_j x} R_{\Delta\lambda y} = \begin{pmatrix} c\Delta\lambda & 0 & -s\Delta\lambda \\ -s\phi_j \, s\Delta\lambda & c\phi_j & -s\phi_j \, c\Delta\lambda \\ c\phi_j \, s\Delta\lambda & s\phi_j & c\phi_j c\Delta\lambda \end{pmatrix} \qquad (4.9)$$

The translation vector then becomes

$$T_{ij} = A_{ij} \begin{pmatrix} 0 \\ 0 \\ h_i + s_i \end{pmatrix} + B_{ij} \begin{pmatrix} 0 \\ 0 \\ d_i \end{pmatrix} - R_{\phi_j x} \begin{pmatrix} 0 \\ 0 \\ d_j \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ h_j + s_j \end{pmatrix} \qquad (4.10)$$

The transformation from tangent plane orientation at site i to earth centered vertical orientation at site j used in the designation mode can now be written

$$(x)_j' = R_{-\delta_j x} A_{ij} (x)_i + R_{-\delta_j x} T_{ij} \qquad (4.11)$$

All the matrices involved are rotations and therefore orthogonal which implies their inverses are given by their transpose. The inverse transformation from earth centered vertical orientation at site j to tangent plane orientation at site i for use in the acquisition mode is therefore given by

$$(x)_i = \left[R_{-\delta_j x} A_{ij}\right]^T \left[(x)'_j - R_{-\delta_j x} T_{ij}\right] \qquad (4.12)$$

where the T superscript indicates the transpose. Simplifying

$$(x)_i = \left[R_{-\delta_j x} A_{ij}\right]^T (x)'_j - A_{ij}{}^T T_{ij} \qquad (4.13)$$

One other relation is useful in calculating the necessary matrices for the ARCAS program which relates $(x)_j$ and $(x)'_i$. Using Eq. (4.11)

$$(x_j) = R_{\delta_j x}(x)'_j = R_{\delta_j x}(R_{-\delta_j x} A_{ij}(x)_i + R_{-\delta_j x} T_{ij})$$

$$= A_{ij}(x)_i + T_{ij} \qquad (4.14)$$

Solving for $(x)_i$

$$(x)_i = A_{ij}{}^T \left[(x)_j - T_{ij}\right]$$

Thus

$$(x)'_i = R_{-\delta_i x}(x)_i = R_{-\delta_i x}\left[A_{ij}{}^T (x)_j - A_{ij}{}^T T_{ij}\right] \qquad (4.15)$$

Therefore, since

$$A_{ji} = A_{ij}{}^T$$

and

$$T_{ji} = -A_{ij}^T T_{ij}$$

all necessary transformations can be calculated from A_{ij} for $i > j$, $R_{-\delta_i x}$ for all i, and T_{ij} for $i > j$.

Assumed Shape of the Earth:

The site latitude, longitude, and height above sea level, ϕ, λ, and h are given from survey data. The angle δ, and distances s and d are calculated from ϕ based on an assumed shape of the earth. The assumed parameters are

$$a = \text{semi major axis} = 6\,975\,459.5 \text{ yards}$$

$$b = \text{semi minor axis} = 6\,951\,973.1 \text{ yards}$$

$$e = \frac{a - b}{a} = \frac{1}{297}$$

The deviation of the normal δ_i is calculated from the relation

$$\tan \delta_i = \frac{e(1 - e/2) \sin 2\phi_i}{1 - (2e - e^2) \sin^2 \phi_i} = \frac{0.003361335 \sin 2\phi_i}{1 - 0.006722670 \sin^2 \phi_i} \tag{4.16}$$

The distances s and d are calculated from

$$s_i = \frac{b^2}{\sin \phi_i (b^2 + a^2 \cot^2 \phi_i)^{1/2}} \tag{4.17}$$

$$d_i = \frac{\cot \phi_i (a^2 - b^2)}{(b^2 + a^2 \cot^2 \phi_i)^{1/2}} \tag{4.18}$$

4.3 Position Updating and Data Validity Checks

An average first difference over five points is computed for use in updating and validity checks. This is given by

$$\Delta x = \frac{x_5 - x_1}{4} = \frac{1}{4} \sum_{i=1}^{4} (x_{i+1} - x_i) \tag{4.19}$$

This linear difference is simple to compute and an example of its expected accuracy is given in Ref. [1]. It is shown there that any error contribution is compatible with the remainder of the system accuracy.

Extrapolation is performed merely by multiplying Δx by the appropriate time delay constant t_d. This is used in the case of both x, y, z and R, A, and E.

That is, for example

$$x_e = x_5 + t_d \Delta x$$

where x_e represents the extrapolated value of the parameter x.

Validity checking is performed in the same way except that the latest value, x_5, is compared against a one-time-interval extrapolated value x_v, based on the previous five points $x_0, x_1, \ldots, x_4$. For x, y, z the test is

$$|x_v - x_5| \leq 3\,\Delta x + \epsilon \qquad (4.20)$$

where ϵ is some small numerical value employed to avoid comparisons against zero. The $\pm\, 3\,\Delta x$ limits allow for extrapolation and random errors while rejecting errors due to lost bits and the like.

Since basic expected random errors in R, A, and E are known, the standard deviation of these distributions may be used in the limits, and the validity check is

$$|R_v - R_5| \leq 4.5\sigma_R \qquad (4.21)$$

Similarly for A and E.

The apparently wide 4.5σ limits are chosen to minimize the probability of rejecting any legitimate data. The object of these validity tests is only to check for reasonable data. That is, one wants to reject data if a significant bit has been lost in the data transmission. However, no real data smoothing or anything of the like is intended in an operation of this kind.

An additional procedure is necessary in the validity check to avoid problems resulting from certain situations which may arise. If, somehow, a legitimate erratic change in course of the vehicle being tracked occurs, a point of data might be rejected. If this occurs, all data from that time on might be rejected and extrapolated values used. This could possibly take place when a missile is sitting on the launch pad and constant position data input is being received which causes the first differences to be zero. At lift off, the change in position could conceivably be significant enough to cause data to be rejected. To avoid this, if a piece of data is rejected, the subsequent two points are not checked but are accepted regardless. The validity check is then again applied to the third following point. Certain fluctuations might result in the computations over five or six cycles if two or three bad pieces of data were received in sequence. However, the probabilities of this are small if the error is just intermittent. The most probable situation of a single lost bit followed by good data is handled properly. If data is bad over a long period of time, the system has most likely failed and nothing can be done in the validity check to take care of this situation.

4.4 Refraction Correction

The equation for use in the refraction correction is as follows and may be derived using Snell's law and assuming a stratified atmosphere along with some simplification obtained by neglecting factors of small effect.

$$\Delta E = - \left[N_s - \bar{N}(z) \right] \cot E (10^{-6}) \tag{4.22}$$

The factor N_s is the surface modulus of the index of refraction and $\bar{N}(z)$ is the average of the modulus of the index of refraction from ground level to the height z of the target. For large altitudes, $\bar{N}(z)$ approaches zero. Some typical values for ΔE from Eq. (4.22) are as follows:

$$E = 12^{\circ} \qquad R = 25 \text{ N.M.} \qquad z \approx 31,500 \text{ ft.}$$

Based on AMR data $N_s \approx 365$ is a typical figure and the average value of N up to a height of about $z = 31,500$ ft. is $\bar{N} \approx 260$. Using Eq. (4.22), $\Delta E \approx -.5$ mils which corresponds favorably to a correction of about -.65 mils (6400 mils = 360 degrees) based on a typical atmosphere given in Ref. 2 .

If $E = 3^{\circ}$, $R = 20$ N.M., then $z \approx 6240$ ft. and $\bar{N} \approx 320$. The correction then is $\Delta E \approx -.9$ mils corresponding to $\Delta E \approx -1.0$ mils given in Ref. [2]. Other calculations yield similar results indicating maximum resulting errors on the order of ± .2 mils and an rms error of somewhat less.

The method for implementing this is based on the fact that the geometry of the tracking situation changes slowly as far as the refraction correction is concerned. Therefore, corrections in the acquisition and designation modes may be the same, and using one computational result from the designation mode to calculate the correction in the acquisition mode will not significantly affect the validity of the refraction correction.

The correction procedure is as follows. First calculate

$$\cot E = \frac{R \cos E}{z}$$

where $R \cos E$ is available from the designation mode and z from the acquisition mode. Then, using z (height) as an argument, a table look-up will be performed on thirteen values to obtain a precomputed value

$$k \left[N_s - \bar{N}(z) \right] 10^{-6}$$

where k is an appropriate scaling constant and $\bar{N}$ is a function of z.

Then calculate

$$\Delta E = -\left(k\left[N_s - N(z)\right] 10^{-6}\right)(\cot E)$$

In the ARCAS program these calculations are most conveniently performed in the acquisition mode with the same ΔE used for the designation mode.

4.5 Scan Mode Selection

Typical scan modes available for a tracking radar are:

I. A spiral scan out to 5 degrees from beamcenter.

II. A raster scan covering 2.4 degrees in azimuth by 8 degrees in elevation with 0.8 degree beam center spacing.

III. A raster scan covering 0.6 degree in azimuth by 8 degrees in elevation with 0.6 degree beam center spacing.

The decision to employ the radar spiral or raster scan mode can be based upon the rate of change of the target azimuth and elevation angles as seen from the acquiring radar. The radar spiral scan mode coverage differs from the radar raster scan mode coverage primarily by the fact that it subtends equal azimuth and elevation angles. An additional asset is that the radar beam begins searching at the center of the scanned area, where the target is most likely located, and spirals outward from the center. Therefore, the radar beam will, in all probability, acquire a target slightly sooner than with the raster scan mode which begins at one corner of the pattern.

Proper selection of the radar scan mode can be automated by computing successive first differences of both the target azimuth and elevation acquisition look angles. The absolute magnitude of the elevation rate $|\dot{El}|$ can be divided by the absolute magnitude of the azimuth rate $|\dot{Az}|$. If the result is less than 2.0 selection of radar scan mode I is automated. If the result is equal to or greater than 2.0, then selection of radar scan mode II or mode III is automated as described below. The decision point of 2.0 is based upon the relative azimuth and elevation coverage ratio of 3.3 for radar scan mode II. This decision point can be modified to reflect current operational practice if desired.

The elevation angles scanned by the two raster modes are identical but the azimuth angles scanned differ by a ratio of 4 to 1. Therefore, the significant parameter is azimuth coverage. A raster scan mode decision based on this and the geometry of the situation can also be readily automated. Automatic computer selection of the best raster scan mode can be based upon the magnitude of the rate of change of azimuth angle $|\dot{Az}|$.

The rate of azimuth angle change that the AN/FPS-16 radar can accommodate, and still maintain reliable target track, is 0 - 750 mils per second. Therefore, the radar raster scan mode can be apportioned between 0 - 750 mils per second on the basis of azimuth coverage as follows.

Radar Scan Mode	Azimuth Coverage of Scan Mode	Weighted Azimuth Coverage of Scan Mode	Azimuth-Rate Coverage
II	2.4°	0.80	600 mils/second
III	0.6°	0.20	150 mils/second

Based on these figures the following raster scan mode selection would be:

Range of Azimuth-Rates	Scan Mode
0 - 150 mils/second	III
150 - 750 mils/second	II

Note that the analysis employed here is not sophisticated and the problem of scan mode selection can be investigated in considerably greater depth. One can consider in much greater detail the actual beam coverage obtained with a certain scan mode, and the rate at which the scan is performed. Probability statements can then undoubtedly be made considering various possible situations which could arise in the geometry of the acquisition problem. The necessary analysis is by no means simple, however, and the selection criteria presented is satisfactory for most practical purposes.

4.6 Coordinate System Selection

The geographical layout of the AMR radar chain makes a coordinate system selection procedure easy to implement in real time. The selection of a coordinate system origin close to the vehicle being tracked is desirable in order to minimize the magnitude of the numbers which are transmitted such that maximum precision is maintained.

The coordinate origin selection is based on the fact that the radar chain lies on approximately a line of 135 degrees azimuth going down range. This approximation is sufficient for the origin selection purposes. Consequently, the barrier lines are 45 degree lines midway between the sites and have equations of the form

$$x = y + b_i \quad , \quad i = 1, 2, 3, 4 \tag{4.23}$$

The b is the x intercept and the slope is unity. For a given site there are then four of these lines with four b intercept values. The procedure is then to compute $\beta = x - y$ and compare with the b_i, $i = 1, 2, 3, 4$. Numbering down range, if $b_{i-1} \leqq \beta \leqq b_i$, then the ith[1] origin is selected.

5. PROGRAM EXTENSIONS

The linear extrapolation technique described in Section 4.3 is extremely well suited for real time calculations. However, future applications might demand greater accuracy for target acquisition and two areas of sophistication enter naturally. The first of these is data smoothing and the second is orbital type calculations. A short description of these two processes appears below.

5.1 Smoothing

The process of data smoothing is performed in order to reduce the effect of random error (noise) in the given data. The mathematical procedure is that of fitting a polynomial (usually second degree) to the observations in the sense of least squares. The reason this procedure is suitable for real time applications is that a "smoothed" point may be represented as a linear combination of the observed points, with the coefficients being only a function of the number of points (if the time interval between points is constant). This same polynomial that is effectively fitted to the data may also be used for extrapolations and therefore the validity check may be constructed around the smoothing procedure. Smoothing is especially important in orbital parameter calculations where the most critical parameter is the velocity vector. Estimating the components of this vector requires as much error to be smoothed out as is practical.

The derivation of the smoothing coefficients is presented below. Note that the analysis treats each coordinate separately which is not quite the same as a three dimensional fit accounting for all coordinates simultaneously. Also, the assumption is implicit that errors are from a stationary random process and uncorrelated.

The assumption is made that a tracking observation x is of the form:

$$x_i = a_0 + a_1 t_i + a_2 t_i^2 + \epsilon_i \qquad (4.24)$$

where ϵ_i is a random error normally distributed with zero mean and variance σ_x^2. That is, a second degree polynomial represents the data correctly, which is a valid assumption in practice. A smoothed point p_i can then be given by:

$$p_i = \sum_{i=0}^{n} b_i x_i \qquad (4.25)$$

where $n + 1$ is the number of points on which the smoothing is performed. A linear combination, which gives a best fit in the least squares sense of this type, is obtained as follows: Let $n = 5$ points for illustrative purposes.

The minimizing condition is

$$S = \sum_{i=0}^{4} (x_i - p_i)^2 = \min \tag{4.26}$$

The a_j, j = 0, 1, 2 are computed by taking partial derivatives and setting results equal to zero. That is, the so-called normal equations are:

$$\begin{aligned} \frac{\partial S}{\partial a_0} &= 2\Sigma(x_i - p_i) = 0 \\ \frac{\partial S}{\partial a_1} &= 2\Sigma t_i (x_i - p_i) = 0 \\ \frac{\partial S}{\partial a_2} &= 2\Sigma t_i^2 (x_i - p_i) = 0 \end{aligned} \tag{4.27}$$

Later calculations are simplified in the case of equally spaced time intervals when the points are relabeled. That is, let i = -2, -1, 0, 1, 2 and upon "normalizing" time one gets $t_i = i$, i = -2, -1, 0, 1, 2. In matrix form the equations become

$$\begin{pmatrix} n & \Sigma t & \Sigma t^2 \\ \Sigma t & \Sigma t^2 & \Sigma t^3 \\ \Sigma t^2 & \Sigma t^3 & \Sigma t^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \Sigma x \\ \Sigma xt \\ \Sigma xt^2 \end{pmatrix} \tag{4.28}$$

or more concisely

$$Ta = x \tag{4.29}$$

Note that by the judicious choice of indices the matrix T has zero elements since $\Sigma t = 0$ and $\Sigma t^3 = 0$. Also, the coefficients a_j will be able to be expressed once and for all in terms of the observed x_i since the t_i are known. To be specific, if α_{ij} are the elements of T^{-1}, then

$$a_0 = \alpha_{11} \sum_{i=-2}^{2} x_i + \alpha_{12} \sum_{i=-2}^{2} x_i t_i + \alpha_{13} \sum_{i=-2}^{2} x_i t_i^2$$

$$a_1 = \alpha_{21} \sum_{i=-2}^{2} x_i + \alpha_{22} \sum_{i=-2}^{2} x_i t_i + \alpha_{23} \sum_{i=-2}^{2} x_i t_i^2 \tag{4.30}$$

$$a_2 = \alpha_{31} \sum_{i=-2}^{2} x_i + \alpha_{32} \sum_{i=-2}^{2} x_i t_i + \alpha_{33} \sum_{i=-2}^{2} x_i t_i^2$$

where

$$\sum_{i=-2}^{2} x_i = (x_{-2} + x_{-1} + x_0 + x_1 + x_2)$$

$$\sum_{i=-2}^{2} x_i t_i = \sum_{i=-2}^{2} x_i i = (-2x_{-2} - x_{-1} + x_1 + 2x_2) \tag{4.31}$$

$$\sum_{i=-2}^{2} x_i t_i^2 = \sum_{i=-2}^{2} x_i i^2 = (4x_{-2} + x_{-1} + x_1 + 4x_{-2})$$

Assume a smoothed value at the midpoint, that is, for $t_0 = 0$, of the given interval is desired. One sets $t_0 = 0$ in the polynomial

$$p_0 = a_0 + a_1 t_0 + a_2 t_0^2 \tag{4.32}$$

from which one obtains

$$p_0 = (\alpha_{11} - 2\alpha_{12} + 4\alpha_{13}) x_{-2} + (\alpha_{11} - \alpha_{12} + \alpha_{13}) x_{-1} + \alpha_{11} x_0$$
$$+ (\alpha_{11} + \alpha_{12} + \alpha_{13}) x_1 + (\alpha_{11} + 2\alpha_{12} + 4\alpha_{13}) x_2 \tag{4.33}$$

giving the smoothed value in the desired form.

The numerical values of these coefficients are in this case

$$b_1 = -.08571430$$

$$b_2 = .34285714$$

$$b_3 = .48571428$$

$$b_4 = .34285714$$

$$b_5 = -.08571430$$

To extrapolate one interval, one only needs to set $t = 3$. The analysis for more points and for higher order equations proceeds in the same manner. A comprehensive discussion and an extensive tabulation of coefficients for many cases appears in Ref. [5].

Note that the implementation on a digital computer requires $n + 1$ multiplies and n adds for smoothing over n points. A highly improved procedure for real time use on a binary digital computer is described in Ref. [3] and termed "almost least squares" smoothing. The procedure is to replace the b_i by β_i given by

$$\frac{\beta_i}{D} = b_i \qquad (4.34)$$

where the β_i are a power of two and D is a normalizing factor. On a binary computer this replaces the multiply and add operations by shift and add resulting in a faster procedure. It has been shown (see Ref. [3]) that using $n + 1$ points in the almost least squares sense is about as efficient as n points in the least squares sense in terms of residual variance.

5.2 Orbital Parameter Calculations

A further refinement in the acquisition problem is to implement position calculations based upon the parameters describing the motion of an orbiting vehicle. These are applicable, and highly desirable, in the case of tracking an orbiting satellite or when tracking a missile in the "free-fall" phase of its flights. Given the orbital parameters, the vehicle's position may be predicted with a high degree of accuracy for a given value of time by solving the equations for that value of time even though the mathematical model is usually highly idealized. Therefore, for a sophisticated extrapolation and acquisition procedure, orbital calculations would be used.

Assume a set of orbital parameters defined as follows, along with a given time t, is available:

- a - semi major axis of the ellipse
- e - eccentricity of the ellipse
- σ - mean anomaly at epoch
- $\cos i$ - where i is the inclination angle
- Ω - right ascension of the ascending node
- ω - argument of the perigee

Typical calculations to obtain position data in the form desired then are as follows. First, Kepler's equation must be solved for the eccentric anomaly E. That is

$$E - e \sin E = n(t + \sigma) \tag{4.35}$$

where

$$n = \sqrt{\frac{\mu}{a^3}} , \quad \mu = \text{constant} \tag{4.36}$$

Note that an iterative procedure is necessary for obtaining E. The true anomaly f is then calculated from

$$f = 2 \tan^{-1} \left[\sqrt{\frac{1 + e}{1 - e}} \tan \frac{(E)}{2} \right] \tag{4.37}$$

and the radial distance by

$$r = a(1 - e \cos E) \tag{4.38}$$

One now computes the coordinates in the orbital plane.

$$\begin{aligned} x^* &= r \cos f \\ y^* &= r \sin f \end{aligned} \tag{4.39}$$

The coordinate system for the orbital plane is then rotated to coincide with an inertial system. This is accomplished by a matrix multiplication.

That is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\omega\cos\Omega - \cos i\sin\Omega\sin\omega & \sin\omega\cos\Omega + \cos i\sin\Omega\cos\omega \\ \cos\omega\sin\Omega + \cos i\cos\Omega\sin\omega & \cos i\cos\Omega\cos\omega - \sin\omega\sin\Omega \\ \sin i\sin\omega & \sin i\cos\omega \end{pmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix} \quad (4.40)$$

A further transformation of coordinates is necessary to reference this x, y, z to the local radar site followed by the conversion to radar coordinates. The above calculations along with those necessary for obtaining a set of orbital parameters from x, y, z position data may be found in Ref. [4].

6. REFERENCES

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