# UNIVERSITY OF ILLINOIS
## DIGITAL COMPUTER

LIBRARY ROUTINE F 2 - 115

| | |
|---|---|
| TITLE | Solution of a System of Differential Equations by Milne's Iterative Method (SADCI Only) |
| TYPE | Closed subroutine |
| NUMBER OF WORDS | Interlude 80 Subroutine 46 |
| TEMPORARY STORAGE | Interlude 0, 1, 2 Subroutine 0, 1<br>13 locations specified by S6 |
| PARAMETERS | S4 - S8 |
| STORAGE LOCATIONS | These parameters and the computed parameter stored in location 9 should not be destroyed during the subroutine. |

| | | | |
|---|---|---|---|
| 4 | 00 F | 00 aF | a is location of first word of auxiliary routine |
| 5 | 00 F | 00 bF | N(b+i) are the independent variables $x_{oi}$ ($i = 0, 1, 2, \ldots, n-1$) |
| 6 | 00 F | 00 cF | Locations c thru c+12 are used as temporary storage |
| 7 | 00 F | 00 nF | n is the number of differential equations to be solved |
| 8 | 00 F | 00 hJ | h is the length of each step of integration |

DURATION

Time of Interlude: $[(11.3 + 3 \sum_{i=0}^{n-1} D_i) n - .1] \times$ maximum $(I_i)$ + .5 milliseconds

Time of Subroutine: $\sum_{i=0}^{n-1} 2.5 I_i + (I_i+1) D_i + 13.5n$ milliseconds per step of integration

n = number of equations

I = number of iterations required per equation

D = duration of auxiliary routine

DESCRIPTION

This routine will solve n simultaneous first order ordinary differential equations expressed explicitly as a function of the variables. It is also possible to solve an equation of order n by reducing the equation to a set of n first order differential equations.

**EXAMPLE**

$y'' + a\, y' + b\, g(x) = 0$ can be written as

$y_0' = y_1$

$y_1' = -a\, y_1 - b\, g(x)$

**PROCEDURE**

At location b, specified by a parameter in location 5, $x_{oi}$ $(i = 0, 1, \ldots, n-1)$ is stored. This is followed by $y''_{oi}$, $y'_{oi}$ and $y_{oi}$. Before the program can be begun it is necessary to have three other values of the function and their derivatives. These are all found by an interlude using several equations devised by W. E. Milne. The information is stored at the following locations:

$N(b + i + h) = y_{-1i}$ $\quad (i = 1, 2, \ldots, n-1)$

$N(b + i + 4h) = y'_{1i}$

$N(b + i + 5n) = y_{1i}$

$N(b + i + 6n) = y'_{2i}$

$N(b + i + 7n) = y_{2i}$

After these values have been found, the subroutine is automatically read in, and the main routine can be begun. Each time the subroutine is called into use it will carry out one integration step of length h and store the value of the function. The subroutine will also find the value of the derivative and store it. These values will be stored in the following manner:

$y'_{ki} = N[b + 2(k + 1)\, n + i]$

$k = 0, 1, \ldots, M$ where M is the number of integration steps

$i = 0, 1, \ldots, n-1$

$y_{ki} = N[b + (2k + 3)\, n + i]$

$k = -1, 0, 1, 2, \ldots, M$

$i = 0, 1, 2, \ldots, n-1$

During the interlude the value of x oscillates from $x_0 + h$ to $x_0 - h$ and then changes to $x_0 + 2h$. Therefore it is necessary for the coder to know where x is at all times. For each equation x can be found at 1S6.

Milne's method uses two quadrature formulas which first predict a value of the function and then make a correction. When the correction is less than or equal to $2^{-39}$, then the new value will be accepted. Since the programmer does not know exactly how many iterations are necessary each time, a counter has been provided which will tell the coder which equation is being used. In location S6, during both the interlude and subroutine $i \times 2^{-39}$ will be found.

It is possible that h has been chosen too large so that the predicted value will be of little use. If this is true, then a carriage return and line feed character will be punched followed by several F's and then the machine will be stopped. Dividing h by 2 will decrease the error by a factor of thirty-two.

During the entire operation the absolute value of the derivative must be less than one-half. If the derivatives are greater than or equal to one-half then overflow will occur and this will also cause F's to be punched. Finally, the punching of F's may also indicate that a faulty subroutine has been written. Before using this code, the programmer should determine by using a code check the average number of iterations necessary. Of course, the number of iterations will vary with the difficulty of the equation.

DISCUSSION OF MATHEMATICAL METHOD USED

A. Interlude

Trial values of $y'_1$ and $y'_{-1}$ are computed from the relations

$y'_1 = y'_0 + hy''_0$ , $y'_{-1} = y'_0 - hy''_0$ (Euler's method)

These values are substituted into equations $A_1$ and $A_2$ to give first approximations to $y_1$ and $y_{-1}$. The first approximations are then substituted into the given differential equation and improved values of $y'_1$ and $y'_{-1}$ are computed. The process is continued until the change in $y'_1$ and $y'_{-1}$ is not more than $\pm 2^{-39}$.

The next step is to use equations B and C in a similar way to obtain a value of $y_2$, the tolerance again being $\pm 2^{-39}$. Here we obtain $y_2$ from B, $y'_2$ from the differential equation and $y_2$ again from C.

$(A_1)\quad y_1 = y_0 + h/24\ (y'_{-1} + 16\ y'_0 + 7\ y'_1) + y''_0\ h^2/4$

$(A_2)\quad y_{-1} = y_0 - h/24\ (7\ y'_{-1} + 16\ y'_0 + y'_1) + y''_0\ h^2/4$

$(B)\quad y_2 = y_0 + 2h/3\ (5\ y'_1 - y'_0 - y'_{-1}) - 2\ y'_0\ h^2$

$(C)\quad y_2 = y_0 + h/3\ (y'_0 + 4\ y'_1 + y'_2)$ (Simpson's Rule)

**B. Subroutine**

With the initial values and the three values found by the interlude an approximation $y_3^{(1)}$ to $y_3$ is made by using formula (D). This value is then substituted into the differential equation and the result is used in formula (E) to obtain an approximation $y_3^{(2)}$ to $y_3$. It should be noted that (E) is simply Simpson's rule. If $y_3^{(1)}$ and $y_3^{(2)}$ agree, then we are finished; otherwise, with $y_3^{(2)}$ an improved evaluation of $y'_3$ is made and the process is continued.

$(D)\quad y_{n+1} = y_{n-3} + 4h/3\ (2y'_{n-2} - y'_{n-1} + 2y'_n)$

$(E)\quad y_{n+1} = y_{n-1} + h/3\ (y'_{n-1} + 4y'_n + y'_{n+1})$

It can be shown that the final correction is a factor of the original correction $C_0$. That is, $C_m = \theta^m C_0$ where $\theta = hy'/3$.

The process will converge if and only if $|\theta| < 1$. Dividing h by 2 will not necessarily reduce the iterations by a factor of two. The number of iterations per step of integrations will be reduced by a factor of two or more only if $\theta \geq 1/2$.

It is now possible to estimate the approximate number of iterations necessary per step of integration. The final correction is never greater than $2^{-39}$ and the original correction is never greater than $2^{-12}$ or else the machine will stop and punch an F.

C. Error

The truncation error can be estimated by the computation.

From Equation (D) $y = y_{n+1}^{(1)} + E_1 , E_1 = 28/90 \, h^5 y^{(5)}$

(E) $y = y_{n+1}^{(2)} + E_2 , E_2 = - h^5/90 \, y^{(5)}$

$$y_{n+1}^{(1)} + y_{n+1}^{(2)} = - 29h^5/90 \, y^{(5)} = 29 \, (- 1/90 \, h^5 \, y^{(5)}) = 29 \, E_2$$

$$E_2 \doteq (y_{n+1}^{(1)} - y_{n+1}^{(2)}) / 29$$

It is also possible to make an estimation of the maximum accumulated error of a numerical integration over a range of length L with N equal steps. Let G be a positive constant such that $|y''| < G$ and let M be a positive constant such that $|y^{(5)}| < M$, then $E_N < L^4 M \, (e^{2LG-1})/180 \, N^4 \, G$.

The value of G can be estimated from the computation. That is, $G \sim (y_n' - y_{n-1}')/(y_n - y_{n-1})$ , $y^{(5)}$ can be estimated by knowledge of the trucation error. Since $E_2 = (-h^5 \, y^{(5)})/90$, $y^{(5)} = (-90 \, E_2)/h^5$. Naturally, the error of the subroutine will be dependent on the error of the values found by the interlude. For complete analysis of the error, see Richter, W. "Sur l'Erreur commisse dans la methode d'integration de Milne", Comptes Rendus de l'Acedemie des Sciences, vol. 233 (1951) pp. 1342 - 1344.

DATE October 26, 1953 RT: 1/23/59
CODED BY Gene H. Golub
APPROVED BY J. P. Nash

lgr

| LOCATION | ORDER | | NOTES PAGE 1 |
|---|---|---|---|
| | 00 K(F2) | | |
| | 26 1000N | | |
| 0 | 00 F | | |
| | 00 3F | | |
| 1 | L5 1S6 | | |
| | 40 S5 | | |
| 2 | 00 F | | |
| | 00 L | | |
| 3 | 38 F | | |
| | 00 F | | 7/16 |
| 4 | 51 8F | | |
| | 66 L | | |
| 5 | S5 S7 | | |
| | 40 9F | | $h/3 \rightarrow 9F$ |
| 6 | 41 S6 | | |
| | 41 F | from 76L | |
| 7 | L5 S5 | | Bring out $x_{0i}$, $y''_{0i}$, $y'_{0i}$, and $y_{0i}$ and |
| | 40 1S6 | | |
| 8 | 19 38F | | store at 1S6, 2S6, 3S6, 4S6. |
| | L4 7L | | |
| 9 | 42 7L | | |
| | L4 5L | | |
| 10 | 46 7L | | |
| | 19 1F | | |
| 11 | L4 F | | Count |
| | 40 F | | |
| 12 | 36 7L | | |
| | 50 8F | | |
| 13 | 75 2S6 | | $y_0 h$ |
| | 40 F | | |
| 14 | L4 3S6 | | |
| | 40 5S6 | | $y'_0 + hy''_0$ |
| 15 | L1 F | | |
| | L4 3S6 | | |
| 16 | 40 10S6 | | $y'_0 - hy''_0$ |
| | 50 F | | |

| LOCATION | ORDER | | NOTES PAGE 2 |
|---|---|---|---|
| 17 | 7J 8F | | |
| | 10 2F | | |
| 18 | 40 9S6 | | $y_0'' \ h^2/4$ |
| | L5 1S6 | | |
| 19 | L4 8F | | |
| | 40 1S6 | | $x_0 + h$ |
| 20 | 49 11S6 | | Store 1/2 at 11S6 |
| | L5 10S6 | | |
| 21 | 10 4F | | Form $y_0 + h/24\ (y'_{-1} + 16\ y'_0 + 7\ y'_1) +$ |
| | L4 3S6 | | $(y_0''\ h^2)/4$ * |
| 22 | 40 F | | |
| | 50 5S6 | | |
| 23 | 7J 3L | | |
| | L4 F | | |
| 24 | 40 F | | |
| | 50 F | | |
| 25 | 75 9F | | |
| | 00 1F | | |
| 26 | L4 9S6 | | |
| | L4 4S6 | | |
| 27 | 40 (6S6) | by 42, 44 | |
| | 50 27L | | |
| 28 | 26 S4 | | Call in auxiliary subroutine |
| | 40 F | | |
| 29 | L0 5S6 | | |
| | 40 1F | | |
| 30 | 19 38F | | |
| | L2 1F | | |
| 31 | 36 32L | | Test $2^{-39} - \lvert y_1'^{(2)} - y_1'^{(1)} \rvert$ * |
| | 41 11S6 | | if $< 0$ |
| 32 | L5 10S6 | | |
| | 40 5S6 | | $y'_{-1} \rightarrow y'_1$ * |
| 33 | L5 F | | |
| | 40 10S6 | | $y'_1 \rightarrow y'_{-1}$ * |

* $y'_1$ and $y'_{-1}$ are interchanged alternatively.

| LOCATION | ORDER | NOTES | |
|---|---|---|---|
| 34 | 41 12S6 | | |
| | 50 12S6 | | |
| 35 | L1 8F | | |
| | 40 8F | | |
| 36 | 00 1F | | |
| | L4 1S6 | | |
| 37 | 40 1S6 | | $x_o \pm h$ |
| | L1 9F | | |
| 38 | 40 9F | | $\pm h/3$ |
| | L1 L | | |
| 39 | 40 L | | Binary switch |
| | 32 41L | | |
| 40 | L5 13L | | |
| | 46 27L | | |
| 41 | 22 20L | | |
| | L5 79L | | |
| 42 | 46 27L | | |
| | L3 11S6 | | Test for zero |
| 43 | 36 20L | | |
| | L5 1S6 | | |
| 44 | L4 8F | | $x_o^{\cdot} + 2h$ |
| | 40 1S6 | | |
| 45 | L5 3S6 | | |
| | L4 10S6 | | |
| 46 | 10 3F | | |
| | 40 F | | |
| 47 | L5 5S6 | | |
| | 10 1F | | |
| 48 | 40 1F | | Form $y_o + (2h)/3\ (5\ y_1' - y_o' - y_{-1}') - 2y_o'' h^2$ |
| | 10 2F | | |
| 49 | L4 1F | | |
| | L0 F | | |
| 50 | 40 F | | |
| | 50 F | | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 51 | 75 9F | | |
| | 00 1F | | |
| 52 | L0 9S6 | | |
| | 00 3F | | |
| 53 | L4 4S6 | | |
| | 40 8S6 | | |
| 54 | 22 54L | from 64 | |
| | 50 54L | | |
| 55 | 26 S4 | | Auxiliary routine |
| | 40 7S6 | | |
| 56 | L4 3S6 | | |
| | 10 2F | | |
| 57 | L4 5S6 | | |
| | 40 F | | Form |
| 58 | 50 F | | $y_0 + h/3\ (y'_0 + 4\ y'_1 + y'_2)$ |
| | 75 9F | | |
| 59 | 00 2F | | |
| | L4 4S6 | | |
| 60 | 40 F | | |
| | L0 8S6 | | $y_2^{(2)} - y_2^{(1)}$ |
| 61 | 40 1F | | |
| | L5 F | | |
| 62 | 40 8S6 | | |
| | 19 38F | | |
| 63 | L0 1F | | $2^{-39} - \lvert y_2^{(2)} - y_2^{(1)} \rvert$ |
| | 36 65L | | if > 0 |
| 64 | L5 8S6 | | if < 0 |
| | 22 54L | | |
| 65 | L5 1S6 | | |
| | 40 S5 | | |
| 66 | 19 18F | | |
| | L4 65L | | Store values permanently |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 67 | L4 7F | | |
| | 40 65L | | |
| 68 | 19 2F | | |
| | L4 12S6 | | |
| 69 | 40 12S6 | | |
| | 36 65L | | |
| 70 | 19 38F | | |
| | L4 S6 | | |
| 71 | 40 S6 | | |
| | L0 7F | | |
| 72 | 32 76L | | Does i = n? |
| | L5 S6 | | If i $\neq$ n change addresses in 65 and 66 |
| 73 | L4 1L | | |
| | 40 65L | | |
| 74 | 00 20F | | |
| | 46 7L | | |
| 75 | L5 36L | | |
| | 42 7L | | |
| 76 | 22 6L | | |
| | L5 5F | from 72 | Find location of $y_{-1}$ and store at 3 |
| 77 | L4 7F | | |
| | 40 3F | | |
| 78 | 50 2L | | |
| | 26 999F | | |
| 79 | 00 6S6 | | |
| | 26 4L | | |
| | 26 1N | | |
| 0 | S5 F | | |
| | L4 14L | | Set link address |
| 1 | 42 44L | | |
| | 41 S6 | | |
| 2 | L5 S5 | | |
| | 40 1S6 | | Bring out $x_{ki}$ |
| 3 | 19 5F | | |
| | 40 9S6 | | |

| LOCATION | ORDER | | NOTES PAGE 6 |
|---|---|---|---|
| 4 | L5 S3 | | Bring out initial values and store |
| | 40 2S6 | | |
| 5 | 19 38F | | |
| | L4 4L | | |
| 6 | 42 4L | | |
| | L4 32L | | |
| 7 | 46 4L | | |
| | L5 9S6 | | |
| 8 | L4 9S6 | | |
| | 40 9S6 | | Count |
| 9 | 36 4L | | |
| | L5 4L | | |
| 10 | 10 20F | | |
| | L4 7F | | Set store address for $y'_{ki}$ and $y_{ki}$ |
| 11 | 42 33L | | |
| | L4 7F | | |
| 12 | 42 31L | | |
| | L1 5S6 | | |
| 13 | 10 1F | | |
| | L4 3S6 | | |
| 14 | L4 7S6 | | |
| | 40 1F | | Form |
| 15 | 50 1F | | $y_{n-3} + (4h/3)\ (2\ y'_{n-2} - y'_{n-1} + 2\ y'_{n})$ |
| | 75 9F | | |
| 16 | 00 3F | | |
| | L4 2S6 | | |
| 17 | 40 8S6 | | |
| | 50 17L | | |
| 18 | 26 S4 | | Call in auxiliary subroutine |
| | L4 5S6 | | |
| 19 | 10 2F | | |
| | L4 7S6 | | |
| 20 | 40 F | | Form $y_{n-1} + (h/3)\ (y'_{n-1} + 4\ y'_{n} + y'_{n+1})$ |
| | 50 F | | |
| 21 | 75 9F | | |
| | 00 2F | | |

| LOCATION | ORDER | | NOTES |
|---|---|---|---|
| 22 | L4 6S6 | | |
| | 40 1F | | |
| 23 | L0 8S6 | | |
| | 40 F | | Store $y_{n+1}^{(2)} - y_{n+1}^{(1)}$ |
| 24 | L5 9S6 | | |
| | 32 28L | | Test for first iteration |
| 25 | 19 14F | | |
| | L2 F | | |
| 26 | 36 28L | | Is $\lvert y_{n+1}^{(2)} - y_{n+1}^{(1)}\rvert > 2^{-12}$ |
| | 92 129F | | No, transfer to 28 and continue |
| 27 | 92 898F | | Yes, print F's and stop |
| | OF F | | |
| 28 | 41 9S6 | | |
| | 19 38F | | Is $\lvert y_{n+1}^{(2)} - y_{n+1}^{(1)}\rvert \leq 2^{-39}$ |
| 29 | L2 F | | |
| | 36 31L | | Yes, transfer to 31 |
| 30 | L5 1F | | |
| | 26 17L | | No, transfer to 17 |
| 31 | L5 1F | | |
| | 40 ( )F | by 12 | Store $y_{n+1}$ |
| 32 | 50 S7 | | |
| | 50 32L | | |
| 33 | 26 S4 | | Call in auxiliary subroutine |
| | 40 ( )F | by 11 | Store $y'_{n+1}$ |
| 34 | 19 38F | | |
| | L4 S6 | | |
| 35 | 40 S6 | | Increase i |
| | L0 7F | | i = n? |
| 36 | 32 40L | | Yes, transfer to 40 |
| | 51 S6 | | No, increase addresses by 1 and continue |
| 37 | 00 59F | | to solve other differential equations |
| | L4 45L | | |
| 38 | 40 4L | | |
| | 19 18F | | |
| 39 | L4 2L | | |
| | 40 2L | | |

| LOCATION | ORDER | NOTES | |
|---|---|---|---|
| 40 | 26 2L | from 36 | |
| | 51 7F | | |
| 41 | 00 60F | | Increase addresses by 2n |
| | L4 45L | | |
| 42 | 40 4L | | |
| | 40 45L | | |
| 43 | L5 44L | | |
| | 46 2L | | |
| 44 | 50 S5 | | Constant |
| | 22 ( )F | by 1 | Leave subroutine |
| 45 | L5 S3 | | |
| | 40 2S6 | | |