UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE Aux. E6-221

TITLE	Integration over a Single Interval (Tabulated Values)
TYPE	Closed subroutine
NUMBER OF WORDS	32 (see Note)
TEMPORARY STORAGE	0, 1
ACCURACY	quadrature (see Method)
DURATION	3.7 + 1.6s msec.
DESCRIPTION	This routine computes an approximation to the integral

$$\int_{a_p}^{a_p + w} f(x)\, dx/w$$

when $f(x)$ is tabulated for the $(2s + 2)$ values, $x =$ $a_p - sw$, $a_p - (s-1)w$, - - -, $a_p - w$, a_p, $a_p + w$, - - -, $a_p + sw$, $a_p + (s + 1)w$. The order of the integration process and therefore its accuracy, for given interval w and function $f(x)$, is controlled by the integration parameter s. The location of $f_k = f(a_p + kw)$ in the memory must be $(p + k)$ where p is arbitrary and $| f_k | < 1/2$. The table of $f(x)$ must extend at least s intervals to the left and to the right of any interval over which $f(x)$ is to be integrated.

The two parameters, p and s, appear in the entry to the subroutine as follows:

	50 p F
q	50 q F
	26 ---
q + 1	00 s F

Control is returned to the left hand side of $(q + 2)$ with the result, $\int_{a_p}^{a_p + w} f(x)\, dx/w$, in A and also in location 0. The parameters a_p and w need not be specified since the result is unchanged by a linear transformation of the variable x.

Since the subroutine permits efficient numerical integration with evenly spaced ordinates, it is specially useful when the interval w is large because of prohibitive cost of computing ordinates. Also, because s appears in the entry, the subroutine can be used for calculations with several values of s chosen by another routine, as required for example when integrations must be repeated until the error is reduced below a pre-set value.

Such an application might occur, for example, during the calculation of an indefinite integral.

NOTE

The 10 constants needed for calculations with s = 0, 1, 2, 3 are stored at 22L, 23L --- 31L in the subroutine. If a value of $s = s_1 > 3$ is to be specified in the entry then the corresponding $(s_1 + 1)$ constants must be placed at relative positions $(u_1 + 22)$ L, $(u_1 + 23)$L, - - - $(u_1 + 22 + s_1)$L where $u_1 = s_1(s_1 + 1)/2$. If the largest value of s needed, say $s = s_2 < 3$, then the subroutine can be shortened by omitting constants at the end of the subroutine.

METHOD

The numerical integration formula* used is

$$\int_a^{a+w} f(x)\,dx/w = \sum_{i=0}^{s} A_{is}\,(f_{-i} + f_{i+1}) + \epsilon(s,w) = g_s + \epsilon(s,w)$$

The quadrature error term $\epsilon(s,w)$ corresponds to approximating the function f(x) by a polynomial of degree (2s + 1) and can be estimated from

$$|\epsilon(s,w)| \leq (w/2)^{2s+2}\,|\,f^{(2s+2)}(a+\xi)\,|\ ,\ |\xi| \leq w$$

The coefficients A_{is} can be generated from the equation

$$g_s - g_{s-1} = 2\,P_s \cdot \mu\delta^{2s} f_{1/2}$$

where $\mu\delta^{2s} f_{1/2}$ is the mean centered difference of degree 2s at $x = a + w/2$ and

*Integration of the Newton-Bessel interpolation formula (Whittaker and Robinson (2nd Ed.), p. 40) gives a quadrature formula that can be expressed in terms of mean central differences (loc. cit., p. 147). In the formula used by the subroutine, the central differences have been replaced by appropriately weighted sums of ordinates.

$$P_s = [2^{2s+1}(2s)!]^{-1} \int_0^1 (z^2 - 1)(z^2 - 9)\cdots[z^2 - (2s-1)^2]\, dz$$

Note that $A_{ss} = P_s$.

EXAMPLE

The error in calculating $\int_a^{a+w} \sin x\, dx/w$ with this subroutine, for $a = 55\,\pi/180$, $w = 15\,\pi/180$ and $s = 3$, is $6.6\text{x}10^{-9}$. The error estimated by the above formula is $|\epsilon(3, 15\,\pi/180)| = 7.6\text{x}10^{-8}$.

DATE	October 15, 1956
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LOCATION	ORDER	NOTES PAGE 1
0	41 1F	
	K5 F	
1	42 4L	
	46 11L	Set addresses
2	L0 20L	
	10 20F	
3	42 11L	
	F5 4L	
4	42 18L	Plant link
	L5 F	
5	42 1F	
	41 F	
6	50 1F	Set addresses
	75 1F	
7	S5 F	
	L4 1F	
8	10 1F	
	L4 21L	
9	42 12L	
	L4 1F	
10	42 19L	Set end constant
	36 18L	End test
11	L5 F	
	L4 F	
12	40 1F	$f_{-i} + f_{i+1}$ to 1
	50 L	
13	7J 1F	$\sum A_{is} (f_{-i} + f_{i+1})$ to 0
	L4 F	
14	40 F	
	L5 11L	
15	L4 20L	
	40 11L	Change addresses
16	F5 12L	
	42 12L	
17	F0 19L	Prepare for end test
	22 10L	

LOCATION	ORDER	NOTES
18	L5 F	Sum to A
	26 F	Link
19	40 1F	
	50 L	
20	LL 4095F	Constants
	00 1F	
21	80 F	
	00 22L	
22	40 F	
	00 0000 0000 0000J	A_{00}
23	40 F	
	00 0416 6666 6667J	A_{01}
24	NO F	
	00 4583 3333 3333J	A_{11}
25	40 F	
	00 0569 4444 4444J	A_{02}
26	NO F	
	00 4354 1666 6667J	A_{12}
27	00 F	
	00 0076 3888 8889J	A_{22}
28	40 F	
	00 0648 3961 6402J	A_{03}
29	NO F	
	00 4212 0535 7143J	A_{13}
30	00 F	
	00 0155 3406 0847J	A_{23}
31	NO F	
	00 4984 2096 5608J	A_{33}