# Division in a One's-Complement Computer

## 1. INTRODUCTION

In a paper entitled "Arithmetic in a One's-Complement Computer" P. Billingsley gives some sufficient conditions in order that a One's-complement computer produce the correct* quotient and correct non-negative remainder. We shall prove that if his conditions are slightly sharpened they are necessary and sufficient. Also we shall prove a conjecture of Campaigne and Gingerich on a necessary and sufficient condition for correct division.

* See condition D, paragraph 4 below.

## 2. THE ALGORITHM

Using the notation of Billingsley the algorithm for division follows:

Initial contents

QR is cleared

XR contains divisor X

AC contains dividend A

Algorithm

1. Case I: If $a_{47} = 0$, go to step 2.

   Case II: If $a_{47} = 1$, absolute subtract XR into AC.

2. Shift AC 24 places. (All shifts are from right to left.)

3. Do 24 times:

   a. Shift AC one place.

b. If $a_{24} = X_{23}$, subtract XR into AC and insert 1 in $q_0$.

If $a_{24} \neq X_{23}$, add XR into AC.

c. Shift QR one place.

Intermediate contents

QR contains $q = q_{23}\, q_{22} \cdots q_0$

XR contains divisor X

AC contains preliminary remainder r.

4. Do one of the following:

a. If $a_{24} = 1$, if Case I: Absolute add XR into AC, replace $q_0$ by $q_{23}$.

b. If $a_{24} = 1$, if Case II: Absolute add XR into AC.

c. If $a_{24} = 0$, if Case I: No action.

d. If $a_{24} = 0$, if Case II: Replace $q_0$ by $q_{23}$

Final contents

QR contains quotient Q

XR contains divisor X

AC contains remainder R

Observe that in 24 repitions of step 3b of the algorithm we successively and in this order determine $q_0, q_{23}, q_{22}, \ldots, q_1$. We can therefore immediately state

LEMMA 1

If A and X are of same sign $q_0 = 1$ and if A and X are of opposite sign $q_0 = 0$.

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An examination of step 4 reveals that $q_{23}$ is not modified. Therefore we can immediately state

LEMMA 2

$q$ and $Q$ are of same sign.

## 3. CONCEPTUAL MODIFICATION OF THE ALGORITHM

In division by hand it is not customary to shift the dividend and successive remainders as is done in steps 3 of the machine algorithm. Instead successively decreasing powers of two times the divisor are added to or subtracted from successive remainders. However, this distinction is only conceptual and we shall throughout this paper treat the machine algorithm as though it takes place in the latter manner.

As an example consider an eight digit AC and four digit QR and XR. Then with X = 0101, A = 00100001 the machine algorithm produces

```
AC    00100001      QR   0000      XR   0101
AL4   00010010

AL1   00100100           0001
- X   00000101
      00011111           0010

AL1   00111110           0010
+ X  -11111010
      01000011           0100

AL1   10000110           0101
- X   00000101
      10000001           1010

AL1   00000011           1011
- X   00000101
      11111101           0111
```

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$q = 0111$ $\qquad$ $Q = 0110$

$r = 11111101$ $\qquad$ $R = 00000011$

The underlined digits are the ones compared with $X_3$ to determine $q_0$, $q_3$, $q_2$ and $q_1$.

Dividing by the second method we have

```
        1011
0101 ⌐00100001
       -0101
       11111000
       + 0101
       00001101
        - 0101
       00000011
         - 0101
       11111101
```

$q = 0111$ $\qquad$ $Q = 0110$

$r = 11111101$ $\qquad$ $R = 00000011$

The underlined digits are again those which are compared and the respective successive remainders in the two divisions are slides of each other.

## 4. CORRECT DIVISION

We shall now state our definition of correct machine division as follows:

DEFINITION

The division algorithm is "correct" if

(i) $A = QX + R$

(ii) $|Q| \leq 2^{23} - 1$

(iii) $0 \leq R < |X|$

LEMMA 3

Division is not correct if $q_0 = q_{23}$.

PROOF

If $q_0 = q_{23}$, then by Lemmas 1 and 2 $Q$ is negative if $A$ and $X$ are of opposite sign. But this contradicts (i) and (iii) of the definition unless $A = 0$ or $Q = 0$. But Billingsley showed $q_0 \neq q_{23}$ if $A = 0$. It only remains to consider $Q = 0$, ie., $A = R$ and therefore by (iii) $A > 0$, $A < |X|$. If $X > 0$ division proceeds as follows:

$$
\begin{array}{r|l}
X & \underline{\quad 1 \qquad\qquad} \\
 & A \\
 & \underline{-2^{23}X} \\
 & A - 2^{23}X = r_1
\end{array}
$$

Thus $q_0 = 1$ and $r_1 < 0$ with $|r_1| \leq (2^{23} - 1)X \leq (2^{23} - 1)^2 < 2^{46}$. Therefore the two left hand digits of $r_1$ are one's and $q_{23} = 0$ and $q_{23} \neq q_0$. The result for $X > 0$ follows by complements.

## 5. THE FUNDAMENTAL THEOREM

Billingsley proved the following:

THEOREM 1

A sufficient condition that a one's-complement computer perform correct division is;

CASE α. $[a_{47} = 0]$

$$A - 2^{23}\ |X| < 0$$

CASE β $[a_{47} = 1]$

$$A - |X| + 2^{23}\ |X| > 0$$

We shall assume the results of Theorem 1 and prove

THEOREM 2

A necessary and sufficient condition that a one's-complement computer perform correct division is;

CASE α $[a_{47} = 0]$

$$A - 2^{23}\ |X| < 0$$

CASE β $[a_{47} = 1,\ (A - |X|)_{47} = 1]$

$$A - |X| + 2^{23}\ |X| \geq 0$$

CASE γ $[a_{47} = 1,\ (A - |X|)_{47} = 0]$

$$1 = 0$$

PROOF OF SUFFICIENCY

CASE α

Follows from Theorem 1.

CASE β

Follows from Theorem 1 unless

$$A - |X| + 2^{23}\ |X| = 0.$$

But then division proceeds as follows if $X > 0$:

$$
\begin{array}{l}
\phantom{X\ |}\;\underline{0\;1\;0\;\ldots\;0\qquad\qquad} \\
X\ |\ A - |X| \\
\quad\underline{+\ 2^{23}\ |X|} \\
\quad A - |X| + 2^{23}\ |X| = 0 \\
\quad\underline{\qquad\qquad -\ 2^{22}\ |X|} \\
\qquad\qquad -\ 2^{22}\ |X| \\
\qquad\qquad\underline{\qquad +\ 2^{21}\ |X|} \\
\qquad\qquad\qquad\vdots \\
\qquad\qquad -2^{22}\ |X| + 2^{21}\ |X| + \ldots + |X| = -\ |X|
\end{array}
$$

The discussion of magnitudes of successive remainders which justifies all additions after the second step is trivial and will be omitted. Then $q = 1\ 0 \ldots 0$ and $r = -\ |X|$. Step 4b implies $Q = q = 2^{23} - 1$ and $R = 0$ which is correct.

The result for $X < 0$ follows by complements.

<u>CASE γ</u> Since the condition is never satisfied there is nothing to prove.

<u>PROOF OF NECESSITY</u>

We shall disregard the case $X = 0$ since in that case the conditions of Theorem 2 are never satisfied and division is never correct.

<u>CASE α</u> Assume that division is correct. Then

$$|A| \leq (2^{23} - 1)\ |X| + R < (2^{23} - 1)\ |X| + |X| =$$
$$= 2^{23}\ |X|$$

CASE [illegible]

To prove that correct division implies $A - |X| + 2^{23}|X| \geq 0$, we shall assume $A - |X| + 2^{23}|X| < 0$ and prove that division is incorrect. Then division proceeds as follows for $X > 0$:

$$\begin{array}{ll} & \quad 0 \\ X & \overline{\left|\, A - |X| \right.} \\ & \quad \underline{+\, 2^{23}\, |X|} \\ & A - |X| + 2^{23}|X| = r_1 \end{array}$$

Here $q_0 = 1$ and $r_1 < 0$ by hypothesis. If $|r_1| < 2^{46}$, $q_{23} = 0$ and by Lemma 3 the division is incorrect. If $|r_1| \geq 2^{46}$,

$$|r| \geq 2^{46} - 2^{22}|X| - \ldots - |X| = 2^{46} - (2^{23} - 1)|X| \geq$$

$$\geq 2^{46} - (2^{23} - 1)^2 = 2^{24} - 1 = 2(2^{23} - 1) + 1 > 2|X|,$$

and $R > |X|$ which is incorrect for division by definition.

If $X < 0$ the result follows by complements.

CASE V

This case is peculiar to machine subtraction which takes $A < 0$ and produces $A - |X| > 0$. But this can happen only if $A$ is a machine number with $a_{47} = 1$ and $a_{46}$ and "several" consecutive digits $= 0$, in other words, if $|A|$ is very large. More precisely if $A < 0$ and $A - |X| > 0$, then $-A + |X| \geq 2^{47}$ and $-A \geq 2^{47} - 2^{23} + 1$. But if $A = QX + R$ with $|Q|$, $|X|$ and $R$ all less than $2^{23}$, then

$$-A \leq (2^{23} - 1)^2 + (2^{23} - 1) = 2^{46} - 2^{23}$$

which is a contradiction.

## 6. THE CAMPAIGNE-GINGERICH THEOREM

Campaigne conjectured condition ( ʃ ) and later Gingerich conjectured conditions ( ˊ ) and ( ¯ ) and indicated a partial proof of

THEOREM 3

Necessary and sufficient conditions that a one's-complement computer perform correct division are that

( ˘ ) $q_0 \neq q_{23}$, and

( $\sigma$ ) $2^{23} |R_1| < 2^{46}$ where $R_1$ is the contents of AC after the first of the 24 steps of 3b in the algorithm.

NOTE. If we think of the dividend as stationary and the successive additions and subtractions of powers of two times X to A as in paragraph 4, then condition ( ˊ ) is that after $q_0$ and before $q_{23}$ have been determined, the remainder $r_1$ satisfies

$$|r_1| < 2^{46}$$

PROOF OF SUFFICIENCY

We shall give a proof by contradiction. That is we shall assume correct division is not performed and shall first show that ( ʃ ) or ( ς ) is contradicted if $X \neq 0$.

CASE [ $a_{47} = 0$ ]

Assume $X > 0$. The proof for $X < 0$ follows by complements.



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Since division is incorrect, by Theorem 1 $A \geq 2^{23} X$.

Then division proceeds as follows:

$$\begin{array}{r|l} X & \overset{1}{A} \\ & \underline{-2^{23}\,|X|} \\ & A - 2^{23}\,|X| = r_1 \geq 0 \end{array}$$

Then the left digit of $r_1 = 0$ and the next digit is zero if and only if $r_1 < 2^{46}$, and then $q_{23} = 1 = q_0$ contradicting $(\delta)$.

CASE $\beta$ $[a_{47} = 1,\ (A - |X|)_{47} = 1]$

The proof is similar to that for case $\alpha$ with $A$, $A \geq 2^{23} X$ and $r_1 = A - 2^{23} X$ replaced respectively by $A - |X|$, $A - |X| + 2^{23}|X| < 0$ and $r_1 = A - |X| + 2^{23}|X|$.

CASE $\gamma$ $[a_{47} = 1,\ (A - |X|)_{47} = 0]$

If $A < 0$ and $A - |X| > 0$, $2^{47} \leq A - |X|$.

Hence

$$r_1 = A - |X| - 2^{23}|X| \geq 2^{47} - 2^{23}|X| \geq$$
$$\geq 2^{47} - 2^{23}(2^{23} - 1) = 2^{46} + 1 > 2^{46}, \text{ and } (\sigma)$$

does not hold.

If $X = 0$, $0 \leq q_0 \equiv a_{47} - 1 \pmod 2$, $0 \leq q_{23} \equiv a_{46} - 1 \pmod 2$ and $r_1 = A$. Then if $a_{47} = a_{46}$, $q_0 = q_{23}$ and $(\delta)$ does not hold, and if $a_{47} \neq a_{46}$, $|A| \geq 2^{46}$ and $(\sigma)$ does not hold.

PROOF OF NECESSITY

We shall prove that if division is performed correctly conditions $(\delta)$ and $(\bar{\sigma})$ hold. But condition $(\rho)$ holds by Lemma 3.

CASE $\alpha$

Assume correct division and not condition $(\delta)$, i.e., $|r_1| \geq 2^{46}$.

Then $|r| \geq 2^{46} - 2^{22}\,|X| - \ldots - |X| =$

$$\geq 2^{46} - (2^{23} - 1)\,|X| \geq 2^{46} - (2^{23} - 1)^2 =$$

$$\geq 2(2^{23} - 1) + 1 \geq 2\,|X| + 1,$$

$|R| > |X|$ and $(\sigma)$ holds.

CASE $\beta$

The proof is identical with Case $(\alpha)$

CASE $\gamma$

No proof is needed since division is never correct.

COROLLARY 1

If $A \geq 0$,

(i) if $A < 2^{23}\,|X|$ division is correct,

(ii) if $2^{23}\,|X| \leq A < 2^{23}\,|X| + 2^{46}$, $(\delta)$ fails and $(\bar{\sigma})$ holds,

(iii) if $A \geq 2^{23}\,|X| + 2^{46}$, $(\delta)$ holds and $(\sigma)$ fails.

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