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DENSITOMETERS AND THEIR USE IN TECHNOLOGY, MEDICINE AND COMMERCE

THE APPLICATION of photography in the fields of engineering, physics, medicine and commerce has expanded the subject of densitometry as it translates photographic results into numerical values. Photographic films are sensitive to radiant energy both below and above as well as in the visible range of the spectrum and since these non-visible radiations affect a photographic emulsion in the same manner as light, it is possible to use photographic means to measure alpha, beta, gamma, X-rays, infrared rays, etc. After development, the resultant density of the film bears a definite relationship to the intensity and duration of the stimulus. Actual calibration of the film can be done by subjecting it to known increments of the stimulus, and after development, a curve can be drawn using the values of the stimulus and the resulting densities of the film.

Figure 1—The Weston Model 877 Photographic Analyzer.

Until a few years ago, densitometers were of the subjective type, that is, the light through the unknown density was balanced against the light through a calibrated step wedge and when the resulting brightnesses were the same, the density of the unknown was equal to the density shown on the calibrated wedge. Densitometers of the subjective type filled a very useful purpose but many people found them difficult to use and prolonged use by an individual became a very trying ordeal. The subjective type densitometers have largely been replaced by the photoelectric type, which have direct reading electrical measuring instruments calibrated in density values. A photoelectric densitometer was described in the June, 1947, issue of WESTON ENGINEERING NOTES and reference can be made to that issue for design details. Figure 1 shows the densitometer referred to and while it is

In This Issue

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not the purpose of this article to discuss densitometers in detail, it does seem pertinent to refer briefly to their calibration. If a collimated beam of light strikes a photographic film, the light passes through the film and emerges as shown in the

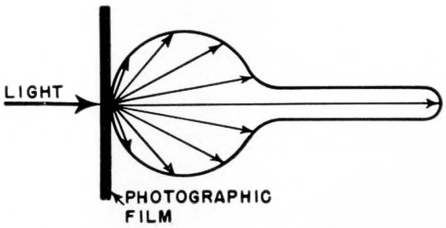

Figure 2—Polar diagram of light emergence.

polar diagram Figure 2. The lengths of the arrows show the magnitude of the luminous flux at the various angles of emergence. The measured density depends upon how the transmitted light is collected. If all of the luminous flux is collected by means of an integrating sphere or a modification thereof, such as the integrating cone shown on the densitometer in Figure 1, then the resulting density is designated as Total or Diffuse Density. If, however, the integrating sphere or integrating cone is located so that it only collects the luminous flux, shown by the long arrow, then the resulting density is referred to as Specular Density. A more complete description of diffuse, specular and intermediate densities can be found in several good photographic textbooks. For our immediate purpose, it can be said that for practically all photographic purposes, the diffuse density values are used. The densitometers referred to in the following applications have been calibrated to measure total light flux or Diffuse Density.

Application of the Densitometer in the Field of Medicine

The use of X-rays to photograph the bone structures and organs of the body is well known, however, the interpretation of the resulting X-ray pictures by the physician or surgeon is often a very difficult task, especially when a series of pictures are taken over a period of weeks or months and it is necessary to know if the diseased area is improving. One method is to place an ivory step wedge in the picture area. This ivory wedge has steps of increasing thickness and, therefore, produces a series of different densities in the X-ray picture. By visually comparing the density of the diseased area with the densities produced by the ivory step wedge, a reference step of equal density can be noted. This can be done on successive X-rays and by always comparing the density of the diseased area with the scale produced by the step wedge, the progress of the disease can be ascertained. One objection to this method of comparing densities is that the visual density of any area is greatly affected by the surrounding areas. For example, if we have two equally gray patches, one of which is surrounded by a lighter background and one of which is surrounded by a darker background, the two gray patches will look entirely different as the gray patch on the lighter background will appear to be much darker than the gray patch on the dark background.

By means of a densitometer, the actual density of the diseased area can be accurately measured. If the X-ray equipment is accurately calibrated and the developing technique highly standardized, then density measurements of the diseased areas will be a direct indication of the condition of the patient. If, however, the X-ray equipment is not accurately calibrated and/or the development cannot be done at a constant time and temperature, then the ivory step should be included and the densitometer used to measure both the density of the diseased area and the steps produced by the ivory wedge. By computing the differences in the density between the same steps on different X-ray pictures, density corrections can be made to compensate for slight differences in exposure and developing technique.

Several hospitals are now using densitometers very successfully and perhaps an actual illustration may be of interest. In one hospital, several physicians reviewed the X-ray pictures of 2,000 patients who had suffered with various kinds of bone diseases. In each case, the set of X-rays was arranged chronologically and the densities of the diseased areas were measured by a densitometer. The density values were plotted against time, and from the resulting curve two things were readily apparent. The direction of the curve showed whether the patient was improving or getting worse and the slope of the curve indicated the rate of change. The interesting thing is that the correlation between the graphs and the condition of the patients was, to quote the physician, "100 per cent." Based upon the above and a few other similar applications, it seems perfectly safe to predict that no X-ray room will in the near future be complete without a densitometer and that by means of carefully controlled exposure and developing technique, the so-called reading of X-ray films will be done by the X-ray technician or nurse by means of a densitometer, and the physician or surgeon will merely look at a

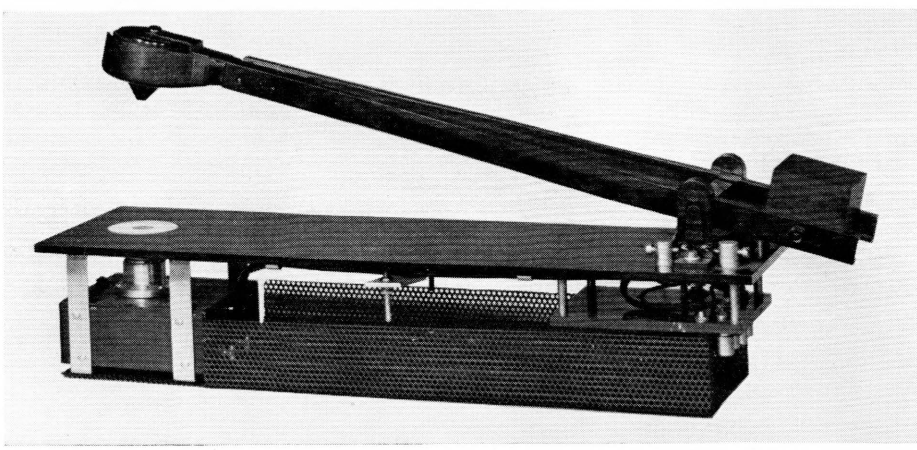

Figure 3—The Weston Model 870 Densitometer is designed to accommodate photographic plates as large as 30 x 40 inches.

graph which will tell him more in a few seconds than he could see in an hour's inspection of the X-ray pictures.

The densitometer, shown in Figure 1, has been used by a number of physicians but because many of the X-ray pictures are rather large, a densitometer with a deeper throat on the photoelectric cell arm is advantageous. Figure 3 shows a densitometer which will accept photographic plates as large as 30 x 40 inches.

Application of the Densitometer in the Field of Health Protection

In this so-called Atomic Age, a great deal of work is being done on fissionable materials and personnel engaged in this field are subjected to the radiations of these radioactive materials. Because of the serious effects these radiations have on the human system, it is very necessary to be sure that certain daily tolerable amounts shall not be exceeded. There are several methods used to monitor these radiations quantitatively but the simplest and most generally used is the photographic scheme. The fact that a radioactive material will act upon a photographic emulsion was discovered by Bequerel in 1896, when he was experimenting with uranium salts. The photographic method used quite extensively today consists of having each person wear a badge containing a piece of dental-type film. Part of the film is covered with a cadmium filter and the other part is not covered except for the paper covering necessary to shield it from visible radiations. The unfiltered portion of the film is sensitive to the alpha, beta and low-energy gamma rays, while the high-energy gamma rays, which are very harmful, will penetrate the cadmium filter and affect the film. After the film has been removed from the badge and developed, the density resulting from these radiations is measured and by means of a previously calibrated curve of Milli-roentgens Exposure against Density, the effective exposure can be ascertained. Figure 4 shows a typical curve. As a matter of interest, a safe daily tolerable exposure is about 50 milliroentgens. It will be noted from the curve that the density resulting from an exposure of 50 milliroentgens is approximately 0.048. Because of this low density, an auxiliary long-scale meter having

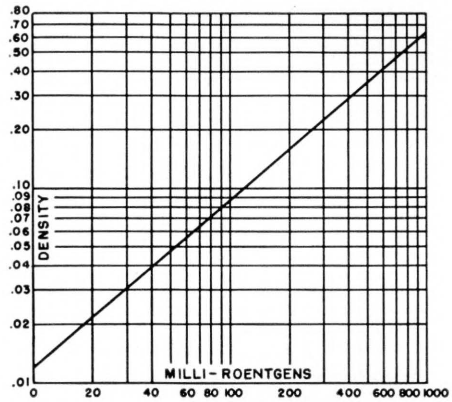

Figure 4—A typical calibrated curve of Milli-roentgens Exposure against Density.

a range of 0 to 1.2 is used with the basic meter which has a range of 1 to 3. The higher-density scale is used to measure the densities of films included in shipments of radioactive materials. Figure 5 shows the densitometer which has been especially designed for the health protection field and many of these are being used in the Armed Services, colleges and universities doing research work and other laboratories who maintain the photographic health monitoring system.

Application of the Densitometer in the Field of Plastics and Paper

Many of the various plastic materials are used as luminaires in conjunction with modern fluorescent lighting and also for decorative work where light transmission is important. Densitometers of special design are being used by both the manufacturer and purchaser to facilitate proper procurement. In the paper industry, densitometers or special modifications of it have been used for several years. Densitometers are usually called Opacity Meters in the paper field but except for the actual scale calibration and the size of the scanning aperture they are virtually the same as the densitometers referred to previously. A typical use of the densitometer or opacity meter is by a large mail order house which uses many tons of catalog paper. Their specifications are very rigid as the opacity of the paper must be sufficient to prevent the printed material on the back from showing through and at the same time the paper must be light in weight to cut excessive mailing costs. The paper manufacturer has a similar opacity meter, hence the rigid specification of the purchaser can be met by the paper manufacturer as the production can be continually checked during its fabrication.

The densitometer shown in Figure 1 has been used in this particular field, however, in order to conform to the accepted practice of the paper industry in having the numerical value of the paper increase with the paper thickness, or the amount of filler used, the scale calibration is in terms of one minus the transmission, which is actually a measure of the light absorption of the paper plus the back surface re-

Figure 5—The Weston Model 877, Type 5 Health Protection Densitometer and its auxiliary long-scale meter.

flection. For example, a thin paper having a transmission of 80% would be rated as 100 – 80 or 20 while a heavier paper having a transmission of 60% would be rated as 100 – 60 or 40.

Application of the Densitometer in the Photo-Mechanical Industries

The photo-mechanical processes of reproducing pictures for printing utilizes the photographic process. While the various photo-mechanical processes are quite different in the specific ways of reproduction, in general, the picture to be reproduced is photographed through a half-tone screen which breaks the picture up into small dots. In photogravure, the half-tone picture is reproduced on a copper plate or cylinder in the form of small hollows. Printing ink is applied and the surface wiped off, leaving the ink in the small hollows. These small hollows, and consequently the ink therein, are almost uniform in surface size but vary in depth according to the strength of tone they represent. After the ink is transferred to the paper, a reproduction of the original picture is then formed by the dots.

Most of the magazines being printed these days are increasing the use of color pictures and the reproduction of these in three or four colors requires the highest kind of craftsmanship in order to obtain balanced separation negatives by means of which color rendition is obtained on the final print. The use of densitometers in the above process allows the entire process to be checked in each step of the process. The photographic plates used in the above process frequently are as large as 30 x 40 inches and the densitometer must be designed to handle these large plates. The densitometer shown in Figure 3 is used by a number of the photo-mechanical plants.

Summary

Densitometers, when used with the photographic process, offer a simple means to reduce the results to numerical values, thus making the test data more concise and clear. In the so-called reading of X-ray pictures, a nurse or technician using a densitometer can interpret the X-ray better than most expert physicians when they rely upon visual inspection. In the Health Protection Field, the densitometer also allows the results to be expressed numerically, thus assuring accurate recording of the monitored data. In case of illness or legal action, the numerical data are available in either a simple table or a graph. In the Photo-Mechanical Field, the densitometer is of inestimable value in determining correct exposures and in checking each step of the process.

E. N.—No. 68 —*A. T. Williams*

METER CALIBRATION WITH THE VARIAC

The following article reprinted by permission of the General Radio Experimenter, *details a very useful arrangement of Variac Transformers for source control and adjustment in calibrating a-c ammeters, voltmeters and wattmeters. The circuit was developed by Professor R. M. Marshall of Purdue University.*

ONE of the many uses of the Variac is in supplying calibrating voltages and currents for electrical indicating instruments such as voltmeters, ammeters, and wattmeters. The accompanying diagrams show a circuit for a-c instrument calibration used by Professor R. M. Marshall of Purdue University.

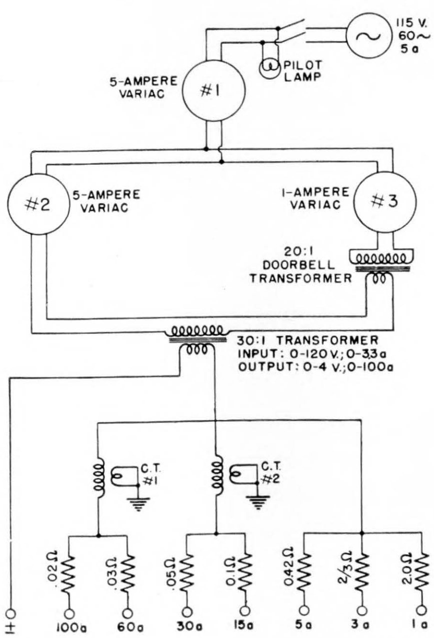

Figure 1—Circuit for current calibrations.

The circuit of Figure 1 is used for current calibrations. The source is a 60-cycle generator, although, if ordinary line voltage variations can be tolerated, ordinary a-c lines can be used. Variac No. 1 is used to adjust the current to give full-scale reading on the instrument to be calibrated, which is connected between the ± terminal and the terminal most nearly representing the full-scale value. With Variacs No. 2 and No. 3 set at maximum, No. 1 is adjusted until the instrument reads full scale or slightly beyond. Zero deflection then corresponds to the zero settings of No. 2 and No. 3. The full range of Variac No. 2 then covers the range of the instrument from zero to full scale. Variac No. 3 changes the current 1/20th as fast as No. 2. The range and precision of control in terms of full-scale deflection are thus the same, regardless of the instrument impedances.

Two current transformers are used so that all tests can be made in terms of a 5-ampere standard in-

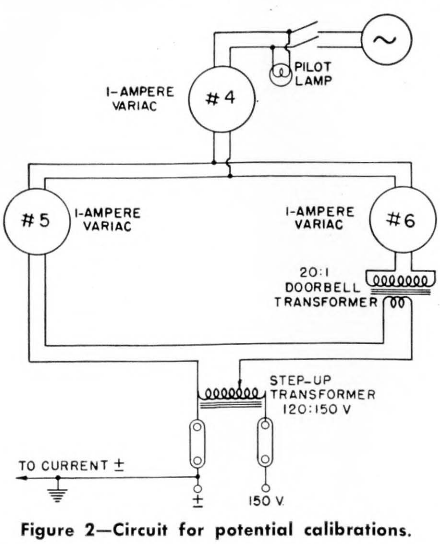

Figure 2—Circuit for potential calibrations.

strument. Each current transformer has several ranges. The standard instrument can be connected to one of the instrument transformers or directly in series with the meter under calibration. The transformers must be shorted as shown, if they are not connected to the standard instrument. The seven resistors are used to stabilize the circuit and to bring the current more nearly into phase with the line voltage. This arrangement gives approximately 100% power factor for wattmeter tests.

The potential circuit of Figure 2 operates in a similar manner. The two terminals provided take care of all instruments rated up to 150 volts. For higher voltages, an additional transformer can be inserted at the points where links are shown. The standard instrument is connected in parallel with the instrument under test.

A grounded connection between the current and potential circuits is provided as shown in Figure 2. When a wattmeter is being tested, the potential terminal leading to the multiplier should be connected to the 150-volt terminal rather than to the + terminal, in order to avoid a high potential between the wattmeter coils.

E. N.—No. 69

THERMAL PROBLEMS RELATING TO MEASURING AND CONTROL DEVICES, PART V—CONTINUED

14. INSTRUMENT POINTER OSCILLATION PRODUCED BY AN ALTERNATING TORQUE IN GENERAL, WITH SPECIAL APPLICATION TO THE EFFECT UPON THE INDICATIONS OF THERMAL INSTRUMENTS, OF THE CYCLIC PULSATIONS IN THE TEMPERATURE OF HEATING CONDUCTORS PRODUCED BY ALTERNATING CURRENT.

Before the equations for the temperature pulsations of a conductor derived in the preceding section can be applied to the problem of determining the resulting oscillations of instrument pointers, it is necessary first to determine the effect of an alternating torque or force in general applied directly to a movable system having inertia, damping, and spring control or its equivalent.

14(a). Alternating Current.

When an alternating current acts upon the movable system of a permanent magnet movable coil type instrument, an oscillating torque is produced which causes the pointer to oscillate above and below its zero position, since the average deflecting torque is zero. If the movable coil carries a direct current, or a rectified alternating current, resulting in a definite deflection, then an alternating current superposed on the direct current or the alternating component of the rectified current, causes pointer oscillation about the mean d-c deflection. The amplitude of the oscillation depends upon the a-c frequency, its magnitude, and the instrument parameters. When, however, an alternating current acts upon the movable system of a square law type instrument, such as electrodynamometers, or movable iron types, then a pulsating torque results which causes the pointer to assume a mean deflection, and to oscillate about this mean deflection at a frequency double that of the alternating current, for the same reason as given in the preceding section for temperature pulsations.

Figure 13 illustrates the oscillating pointer of an instrument through which an alternating or rectified current is passing. The current in the instrument may be applied directly, or it may be the alternating current generated by a thermocouple thermally connected to a heating conductor through which an alternating current is passing, causing a cyclic pulsation in temperature.

The pointer oscillates to an amplitude a above and below a mean deflection D if a direct current is also acting or if the current is a rectified current, or about zero if alternating current only is acting.

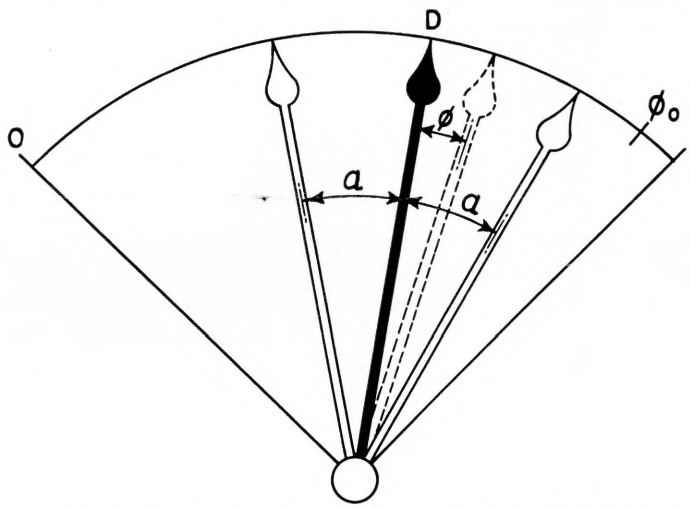

Figure 13—Oscillation of an instrument pointer resulting from the alternating or pulsating torque produced by alternating or rectified current, and specifically by the pulsating current generated in a thermocouple connected to a conductor carrying alternating current which produces cyclic pulsations in temperature.

Permanent Magnet Type Instruments.

It will be shown later that the angular deviation ϕ of an instrument pointer from its mean position at any time, when an alternating torque acts upon its movable system, is

$$\phi = \frac{\phi_o}{\sqrt{4n^2T_o^2f^2 + (1 - T_o^2f^2)^2}} \sin(\omega t - \alpha) \qquad (80)$$

in which

$$\alpha = \sin^{-1}\frac{2n}{\sqrt{4n^2 + \left(\frac{1}{T_of} - T_of\right)^2}} = \cos^{-1}\frac{\frac{1}{T_of} - T_of}{\sqrt{4n^2 + \left(\frac{1}{T_of} - T_of\right)^2}}$$

$\phi_o = F/S$, where F is the crest value of the applied alternating torque and S is the spring stiffness. That is, ϕ_o is the angular deflection which would result if a direct current having a value equal to the crest value of the alternating current were applied to the instrument, assuming that the same relation between deflection and torque at the mean position, held throughout the angular deflection.

T_o = undamped period of the instrument; seconds.

n = specific damping coefficient, = ratio of actual damping coefficient to that required to produce critical damping.

f = frequency of applied alternating torque.

f_o = frequency of the alternating current.

These equations for α show that its sine is always positive; therefore, it may be either in the first or second quadrants only. Again, since T_of is usually greater than 1 in practice, $\cos\alpha$ is usually negative and therefore the angle of lag, α, is usually in the second quadrant and approaches 180° as T_of increases. When $T_of=1$, the angle of lag is 90°.

The maximum angular deviation a from the mean position as derived from Equation (80) is

$$a = \frac{\phi_o}{\sqrt{4n^2T_o^2f^2+(1-T_o^2f^2)^2}}. \qquad (81)$$

Equations 80 and 81 are perfectly general. They are applicable, both for motions of rotation and translation, to any oscillating body having inertia, damping, and a control torque or force proportional to the displacement, and acted upon by a driving torque or force which varies sinusoidally.

If the motion is one of translation, then ϕ_o becomes a distance instead of an angle. When the equations are applied to a permanent magnet type instrument, the torque frequency f is equal to the frequency f_o of the alternating current.

Example: Let us assume a permanent magnet movable coil instrument through which an alternating current is passing, having a frequency of 3 cycles per second and of such a value that a direct current equal to its crest value would produce a scale deflection ϕ_o, of say 1.5 times full scale deflection.

Let the undamped period $T_o=2$ seconds, and the specific damping coefficient $n=0.8$, corresponding to a damping factor of 65. Then the maximum amplitude of oscillation of the pointer from Equation (81), remembering that $f=f_o$, is

$$a = \frac{1.5 \times \text{full scale}}{\sqrt{4\times 0.8^2\times 2^2\times 3^2+(1-2^2\times 3^2)^2}}$$

Then,

$$a = \frac{1.5}{36.2}\times \text{full scale} = 0.0414\times \text{full scale}$$

That is, the amplitude is 4.14 per cent of full scale.

Square Law Instruments (A-C Instruments).

In an instrument operating on the square law principle through which a sinusoidal alternating current having a crest value of I_m is passing, a pulsating torque is produced which may be considered as consisting of a constant value on which is superposed an alternating component having a crest value proportional to $\frac{I^2_m}{2}$, and a frequency twice that of the alternating current, for the same reason as given in Section 12 for temperature pulsation and illustrated in Figure 12.

The constant component of the torque acting, resulting in the mean deflection D, is also proportional to $\frac{I^2_m}{2}$. The amplitude of the pointer oscillation resulting from the torque pulsation at double the frequency of the alternating current, may be computed by Equation 81, by substituting $2f_o$ for f, and D for ϕ_o for the following reason. As the crest value of the a-c component of the torque pulsation, corresponding to F in the definition, is the same as the constant component of the torque which is DS, resulting in the mean deflection D, then $DS=F$, and we have $\phi_o=F/S=D$. Substituting these values in Equation (81) we obtain for the maximum amplitude of a square law type instrument pointer

$$\frac{a}{D} = \frac{1}{\sqrt{16n^2\,T_o^2f_o^2+(1-4T_o^2f_o^2)^2}} \qquad (82)$$

In square law and other non-linear law type instruments the period in general depends upon the scale position of the movable element. Therefore, theoretically at least, the value of T_o should be determined at the deflection D, although in most cases in practice, very little difference will be found throughout the scale.

14(b). Pointer Oscillation for Non-Sinusoidal Wave Shapes.

Square Top Wave.

The Fourier analysis of the square top wave shows that most of the oscillating torque caused by such a wave results from the fundamental component, which has a crest value of $4(I/\pi)$, where I is the amplitude of the square top wave.

If in this case, ϕ_o in Equation (81) is determined as the angle which would result if a direct current equal to I were applied to the instrument, than for a crest value of $4(I/\pi)$,

$$a \cong \frac{(4/\pi)\phi_o}{\sqrt{4n^2T_o^2f_o^2+(1-T_o^2f_o^2)^2}} \qquad (83)$$

If greater accuracy is desired, which seldom is necessary, then the effects of all harmonics can be added for the total effect. The same procedure may be followed for other wave shapes.

Full Wave Rectified Alternating Current.

When full wave rectified alternating current is passed through a permanent magnet movable coil instrument, a pulsating torque is produced which may be considered as consisting of a constant component upon which is superposed an alternating component. The constant component, which is the mean value of the pulsating torque, produces the deflection D, and is proportional to $2(I/\pi)$ where I is the crest value of the alternating current.

The Fourier analysis of this rectified wave shows that the second harmonic is responsible for most of the oscillating torque, and that it has a crest value of 2/3 that of the constant component, that is, the mean value resulting in the deflection D.

Therefore, since by definition, ϕ_o in Equation 81 is equal to the crest value of the oscillating torque divided by the spring stiffness, we have

$$\phi_o = \frac{\text{crest value}}{S} = \frac{2/3 \text{ mean value}}{S} = \frac{(2/3)DS}{S} = (2/3)D$$

Substitute this in Equation 81, and remember that the frequency $f = 2f_o$, then we have for the approximate value of the amplitude of the pointer oscillation for full wave rectified current,

$$\frac{a}{D} \cong \frac{2}{3\sqrt{16n^2T_o^2f_o^2 + (1 - 4T_o^2f_o^2)^2}} \tag{84}$$

15. OSCILLATION OF A POINTER OF AN INSTRUMENT OPERATED FROM A THERMOCOUPLE THERMALLY CONNECTED TO A CONDUCTOR CARRYING AN ALTERNATING CURRENT OR A FULL WAVE RECTIFIED ALTERNATING CURRENT.

Equation 69 in Section 12, gives the relation of the crest value, θ_m, of the alternating component of the temperature pulsating wave, to the mean temperature θ_o, as

$$\frac{\theta_m}{\theta_o} = \frac{1}{\sqrt{1 + 16\pi^2(f_o t_o)^2}}$$

It is also shown that the temperature pulsating wave is exactly the same for both alternating current and full wave rectified alternating current for equal crest values. Therefore, they have the same effect upon pointer oscillation when applied to the same instrument, and the same equations apply to both.

It will be noted that this result differs from that when alternating current and rectified alternating current are applied directly to the indicating instrument instead of by means of heat they generate, for the reason that whereas they produce the same temperature waves, their torque waves differ.

When θ_m and θ_o act upon an instrument by means of a thermocouple, they produce respectively the crest value of the alternating component of the torque, and the mean value of the acting torque. They are therefore proportional to ϕ_o and D respectively from which we may write

$$\frac{\phi_o}{D} = \frac{\theta_m}{\theta_o} = \frac{1}{\sqrt{1 + 16\pi^2(f_o t_o)^2}} \tag{85}$$

Substituting the value for ϕ_o from Equation 85 in Equation 81, remembering that the frequency $f = 2f_o$, we have the maximum amplitude a of the pointer oscillation resulting from temperature pulsations in a conductor

$$\frac{a}{D} = \frac{1}{\sqrt{[1 + 16\pi^2(t_o f_o)^2][16n^2f_o^2T_o^2 + (1 - 4f_o^2T_o^2)^2]}} \tag{86}$$

If $t_o f_o$ is equal to or greater than 1, then with an error of about 0.3 per cent or less, Equation 86 becomes

$$\frac{a}{D} = \frac{1}{4\pi t_o f_o \sqrt{16n^2f_o^2T_o^2 + (1 - 4f_o^2T_o^2)^2}} \tag{87}$$

Example: Compute the amplitude of oscillation relative to the mean deflection of the pointer of the instrument referred to in the example given after Equation 81, in which the undamped period $T_o = 2$ seconds, and the specific damping coefficient $n = 0.8$, but connected to a vacuum type thermocouple having a time constant $t_o = 1.54$ seconds, carrying an alternating current having a frequency of 1 cycle per second. Then from Equation 87 we have

$$\frac{a}{D} = \frac{1}{4\pi \times 1.54 \times 1 \sqrt{(16 \times 0.8^2 \times 1^2 \times 2^2) + (1 - 4 \times 1^2 \times 2^2)^2}}$$

from which $\frac{a}{D} = 0.00316$.

That is, the maximum deviation of the pointer from its mean position during oscillation is 0.316 per cent of its mean deflection D.

16. DERIVATION OF EQUATION 80.

Assume that the movable system of an instrument, having moment of inertia, damping, and spring control or its equivalent, is acted upon by sinusoidal alternating deflecting torque, and it is required to find the angular position of the pointer at any time, and in particular, its maximum amplitude.

Let K = moment of inertia; gm cm².
G = damping coefficient; dyne cm. per radian per sec.
S = spring stiffness; dyne cm. per radian.
n = specific damping coefficient.
F = crest value of alternating driving torque; dyne cm.
ϕ = the angular displacement of pointer at any time t.
ω = angular frequency, radians per sec.
j = $\sqrt{-1}$.

The driving torque at any time t is

$$F \sin \omega t$$

This is opposed and equal to the algebraic sum of the accelerating torque, the damping torque, and the spring torque. Stated mathematically, and in the order given, we have after dividing through by K,

$$\frac{d^2\phi}{dt^2} + \frac{G}{K}\left(\frac{d\phi}{dt}\right) + \frac{S}{K}\phi = \frac{F}{K} \sin \omega t \tag{88}$$

Since we are interested in the steady state only, and as the system is assumed linear, we know that the resulting motion will be sinusoidal. To integrate the equation, therefore, let

$$\phi = \phi_m \sin \omega t \qquad (89)$$

where ϕ_m = the crest value in phase and magnitude, of the pointer motion, which is to be determined.

Put the equation into the exponential form and we have

$$\phi = \phi_m \sin \omega t = \frac{\text{Imaginary Part}}{j}\left[\phi_m \epsilon^{j\omega t}\right] \qquad (90)$$

also

$$\frac{F}{K} \sin \omega t = \frac{I.P.}{j}\left[\frac{F}{K}\epsilon^{j\omega t}\right] \qquad (91)$$

For brevity we shall omit repeating (Imaginary Part)/j and remember to insert it where its use becomes necessary.

Then by differentiating Equation 90,

$$\frac{d\phi}{dt} = j\omega\phi_m\epsilon^{j\omega t} \quad \text{and} \quad \frac{d^2\phi}{dt^2} = -\omega^2\phi_m\epsilon^{j\omega t}$$

Substitute these values in Equation 88, divide through by the common factor $\epsilon^{j\omega t}$ and we have

$$-\phi_m\omega^2 + j\frac{G}{K}\phi_m\omega + \frac{S}{K}\phi_m = \frac{F}{K} \qquad (92)$$

Then,

$$\phi_m = \frac{F/K}{\left(\frac{S}{K} - \omega^2\right) + j\frac{G\omega}{K}}$$

which when rationalized becomes

$$\phi_m = \frac{\frac{F}{K}\left[\left(\frac{S}{K} - \omega^2\right) - j\frac{G\omega}{K}\right]}{\frac{G^2\omega^2}{K^2} + \left(\frac{S}{K} - \omega^2\right)^2} \qquad (93)$$

$$\text{But } \phi = \frac{I.P.}{j}\phi_m\epsilon^{j\omega t} = \frac{I.P.}{j}\phi_m[\cos \omega t + j \sin \omega t] \qquad (94)$$

Substitute in Equation 94 the value for ϕ_m found in Equation 93 and we have

$$\phi = \left(\frac{I.P.}{j}\right)\frac{F/K}{\frac{G^2\omega^2}{K^2} + \left(\frac{S}{K} - \omega^2\right)^2} \times \left[\left(\frac{S}{K} - \omega^2\right) - j\frac{G\omega}{K}\right](\cos \omega t + j \sin \omega t) \qquad (95)$$

Expanding and using imaginary parts only, and dividing through by j we have

$$\phi = \frac{F/K}{\frac{G^2\omega^2}{K^2} + \left(\frac{S}{K} - \omega^2\right)^2}\left[\left(\frac{S}{K} - \omega^2\right)\sin \omega t - \frac{G\omega}{K}\cos \omega t\right] \qquad (96)$$

Multiply and divide by

$$\sqrt{\frac{G^2\omega^2}{K^2} + \left(\frac{S}{K} - \omega^2\right)^2}$$

and Equation (96) reduces to

$$\phi = \frac{F/K}{\sqrt{\frac{G^2\omega^2}{K^2} + \left(\frac{S}{K} - \omega^2\right)^2}} \sin(\omega t - \alpha) \qquad (97)$$

where

$$\alpha = \sin^{-1}\frac{\frac{G\omega}{K}}{\sqrt{\frac{G^2\omega^2}{K^2} + \left(\frac{S}{K} - \omega^2\right)^2}} = \cos^{-1}\frac{\left(\frac{S}{K} - \omega^2\right)}{\sqrt{\frac{G^2\omega^2}{K^2} + \left(\frac{S}{K} - \omega^2\right)^2}}$$

Divide numerator and denominator of Equation (97) by S, rearrange, and we have

$$\phi = \frac{\phi_o}{\sqrt{\frac{G^2\omega^2}{S^2} + \left(1 - \frac{K}{S}\omega^2\right)^2}} \sin(\omega t - \alpha) \qquad (98)$$

where $\phi_o = F/S$

Now, since by definition, $G = nG_o = 2n\sqrt{KS}$, and that the undamped period of the instrument is $T_o = 2\pi\sqrt{\frac{K}{S}}$, then

$$\frac{K}{S} = \frac{T_o^2}{4\pi^2}, \quad \frac{G^2}{S^2} = \frac{4n^2KS}{S^2} = \frac{n^2T_o^2}{\pi^2}, \text{ and } \omega^2 = 4\pi^2 f_o^2$$

Substitute these values in Equation (98) and in the equations for the phase angle α, and we obtain

$$\phi = \frac{\phi_o}{\sqrt{4n^2T_o^2f_o^2 + (1 - T_o^2f_o^2)^2}} \sin(\omega t - \alpha)$$

in which

$$\alpha = \sin^{-1}\frac{2n}{\sqrt{4n^2 + \left(\frac{1}{T_of_o} - T_of_o\right)^2}} = \cos^{-1}\frac{\frac{1}{T_of_o} - T_of_o}{\sqrt{4n^2 + \left(\frac{1}{T_of_o} - T_of_o\right)^2}}$$

which is Equation 80.

E. N.—No. 70 —*W. N. Goodwin, Jr.*

This concludes the subject of instrument pointer oscillation, caused in general by alternating torque, and in particular by temperature pulsations in thermal instruments.

The articles to follow will consider the temperature and the temperature distribution in a conductor heated by current or by radiation.

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