Weston ENGINEERING NOTES

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LUMINANCE MEASUREMENTS ON TELEVISION PICTURE TUBES

EDITORIAL FOREWORD: *This paper points out the rather large errors that may occur when photoelectric meters are used to measure the luminance of television picture tubes, unless the load resistance on the barrier-layer photocell is rather low. Standard photometric practice is to calibrate meters on a steady light source, but if the load, or meter resistance, is high enough to produce a non-linear characteristic, the indicated luminance will be below the true average which the eye sees.*

ALTHOUGH the science of photometry is based upon what the average human eye sees, the characteristics of human eyes vary considerably as shown in the research work by Coblenz and Emerson, as published in the *Bulletin* of the Bureau of Standards, Volume 14, 1918-19, on pages 167 to 234. For many years only visual photometers were available, but due to the vagaries of human eyes, many observers had difficulty in obtaining comparable results under identical conditions. This fact, plus the inordinate time required, has always made photometrists desire a physical means of measurement. At the same time, scientists have realized that such standardization must revert to the visual photometers. With the advent of the barrier-layer selenium type of photoelectric cell, physical photometry took a long step forward and, today, visual photometers are being rapidly superseded by physical photometers of the photoelectric type. By means of a photoelectric illumination meter or a photoelectric luminance meter, a layman can measure illumination or luminance more accurately than a trained photometrist can determine it with visual photometers, and in a small per cent of the time. This ease of measurement, plus the saving of time, has increased the scope in which physical photometers have been used, and it was a natural step to use these same meters for measuring the brightness of picture tubes. However, unless the luminance meters are specially designed for this application, rather large errors will occur due to the following phenomena:

1. The peak brightness of the picture tube is several thousand times the average brightness.
2. When a photoelectric cell is subjected to high brightness and the load resistance is also high, the relationship of photoelectric current to light is non-linear and, consequently, the photoelectric cell is not a good integrator of such light.
3. The human eye does not produce a sensation directly proportional to the light stimulus, but it is a good integrator of light.

Under the above conditions, the physical photometer will not agree with visual measurements and the reason for this is explained below.

Figure 1 shows the effect of load resistance on the current output of a photoelectric cell for various luminance or brightness values. It will be noted that at zero load resistance, the relationship is perfectly linear, while for a higher resistance,

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Luminance Measurements on Television Picture Tubes

Thermal Problems Relating to Measuring and Control Devices—Part VII (Continued)

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the relationship is linear at low brightness values, but very non-linear for high brightness values.

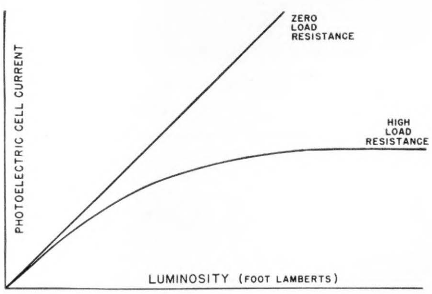

Figure 1—Curves showing effect of load resistance on the photoelectric cell current.

The response of the eye is also non-linear, that is, the sensation does not increase in proportion to the light. Fechner, in 1858, stated that as the stimulus to the eye is increased in geometrical progression, the sensation is only increased in arithmetic progression. Another factor which enters into the problem is the ability of the eye and the photoelectric cell to integrate and average the brightness. The eye is an excellent integrator, as can be shown experimentally, while the photoelectric cell will only integrate correctly if it works into a resistance load which is low enough to maintain linearity. We can control the output linearity of the photoelectric cell by means of the load resistance, and from experimental data, it is evident that a zero resistance load produces the best results. Such an instrument could be readily built, but it would be considerably more expensive than a simple microammeter. If, however, we work the photoelectric cell into a relatively low resistance, we will obtain a satisfactory linear relationship, and the visual brightness and the brightness as indicated on a photoelectric meter will agree to within close limits. This statement is based upon the supposition that the photoelectric meter is calibrated in accordance with accepted photometric practice, that is, on steady light using tungsten lamps of known candlepower, etc. Obviously, if meters are calibrated using the television picture tube as a source, then they will indicate correctly on this source, but they will not indicate correctly on other light sources. By proper design, it is possible to construct luminance meters which will indicate the correct brightness on steady light and on pulsed light as used in television picture tubes.

Figure 2 shows the per cent that the meter will indicate low for various load or meter resistances when used at different brightness values. These curves were plotted from data taken by the writer at the Du Mont Laboratories, and the co-operation of Messrs. Hoagland, Scott and their associates is gratefully acknowledged.

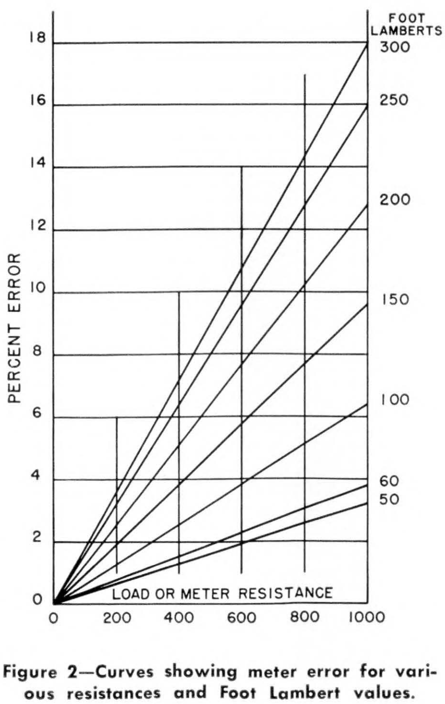

Figure 2—Curves showing meter error for various resistances and Foot Lambert values.

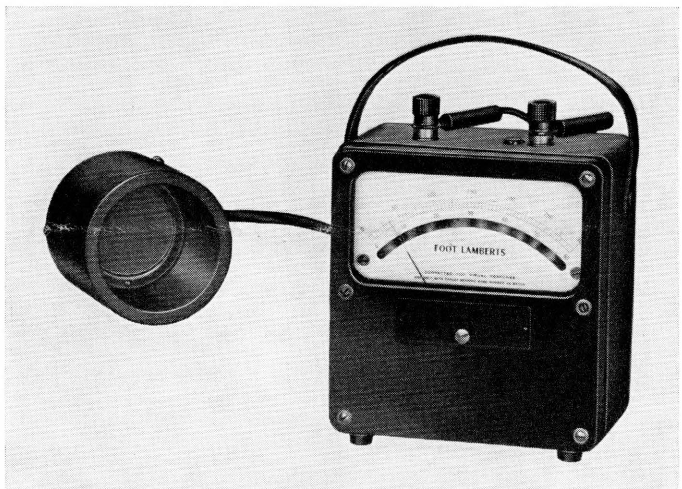

Figure 3—The Weston Model 931 Foot Lambert Meter complete with the photoelectric cell and tubular baffle.

The Weston Model 931 Foot Lambert Meter, shown in Figure 3, has been designed to meet the requirements of the television engineers, as outlined by the Phosphor Committee of the Radio Manufacturers' Association. Actually, the committee tentatively specified ranges of 0-250 and 0-50 Foot Lamberts, but for design reasons the ranges actually are 0-300 and 0-60 Foot Lamberts. The meter is equipped with a visual correction filter which corrects the photoelectric cell to match the accepted visibility curve as specified by the International Commission on Illumination. It is equipped with a tubular baffle which restricts the oblique light, and results in an acceptance as shown in the polar diagram, Figure 4. Because of the restricted acceptance angle, the relatively low transmission of the visual correction filter and the small amount of light available, it is necessary to use a very sensitive microammeter which inherently has a very high critical damping resistance. Because of the high critical damping resistance, it was not possible to arrange the circuit so that the desired ranges would be obtained, and at the same time keep the load resistance low enough to obtain linearity of photoelectric cell current and high enough to maintain good damping characteristics on the microammeter. For this reason, the basic meter has a range of 0-60 Foot Lamberts, and the 0-300 Foot Lambert range is obtained by means of a mechanical filter. The

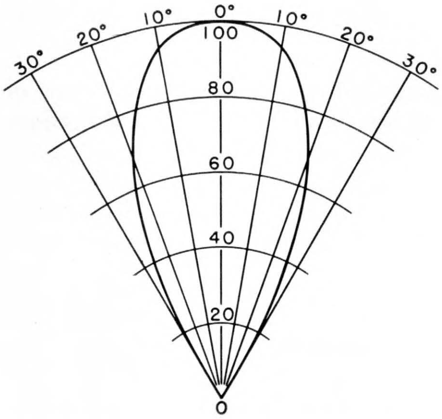

Figure 4—Polar diagram showing relative response of meter to incident and oblique light.

TABLE OF LUMINANCE CONVERSIONS

Unit	*Light Flux Density*	*Conversion Factor**
Foot Lambert	1 Lumen per Sq. Foot	1.000
Lambert	1 Lumen per Sq. Cm.	0.001076
Millilambert	0.001 Lumen per Sq. Cm.	1.076
Apostilb	1 Lumen per Sq. Meter	10.760
Candles per Sq. Inch	1 C.P. per Sq. Inch	0.00221
Candles per Sq. Foot	1 C.P. per Sq. Foot	0.318

* The Weston Model 931 Foot Lambert Meter is calibrated to indicate luminance directly in Foot Lamberts. By means of the conversion factor, the meter indications can be translated into the other units of luminance.
(Meter Indication × Conversion Factor = Units in left-hand column.)

resistance of the 0-60 range will be 360 ohms or less, and from the curves shown in Figure 2, the resultant error due to the pulsed light will be from 0 to 1.4 per cent, depending on the actual Foot Lambert value. Obviously, the 0-300 range will have the same resistance as the 0-60 range, but because of the mechanical filter, which reduces the actual amount of luminous flux which reaches the photoelectric cell, the error due to the pulsed light will be the same as on the 0-60 range.

When using the meter, it is recommended that the photoelectric cell assembly be placed against the face of the tube and then back off until a maximum reading is obtained. It will be noticed that the reading will increase slightly to maximum and then decrease. The maximum reading will be the correct luminance value expressed in Foot Lamberts. The most commonly used unit of luminance in the United States is the Foot Lambert; however, other units are used, both here and abroad, hence, the above table may be of interest.

E. N.—No. 77 —*A. T. Williams*

THERMAL PROBLEMS RELATING TO MEASURING AND CONTROL DEVICES—PART VII (Continued). DISTRIBUTION OF TEMPERATURE WITH TIME ALONG UNIFORMLY HEATED CONDUCTORS CONNECTED BETWEEN TERMINALS

Introduction

IN THE first section of Part VII, the transient temperature distribution along the general conductor was investigated. In this section, a similar study will be made of a special conductor from which the amount of heat lost by convection is negligible relative to that conducted to the terminals. Some practical examples of the use of the equations will be given, together with the derivation of the general equations.

22. GIVEN A CONDUCTOR CONNECTED BETWEEN TERMINALS, HEATED UNIFORMLY AT A CONSTANT RATE, AND FROM WHICH THE HEAT LOST BY CONVECTION TO THE SURROUNDING MEDIUM IS NEGLIGIBLE RELATIVE TO THAT CONDUCTED TO THE TERMINALS. FIND THE TEMPERATURE DISTRIBUTION AT ANY TIME AFTER THE INITIAL APPLICATION OF HEAT.

This case applies in practice to conductors which are very short relative to their cross-sections; to shunts having multiple metal resistance sheets in which the inner sheets have very inefficient contact with the surrounding air, and other similar conductors.

It is desired to determine the temperature elevation, θ, at any point at a distance x from the terminal T_1, along the conductor connected between terminals as shown in Figure 17, reproduced from Section 21 for convenience, at any time, t, after the initial application of the heat. It is assumed that the terminals have the same temperature as the surrounding medium.

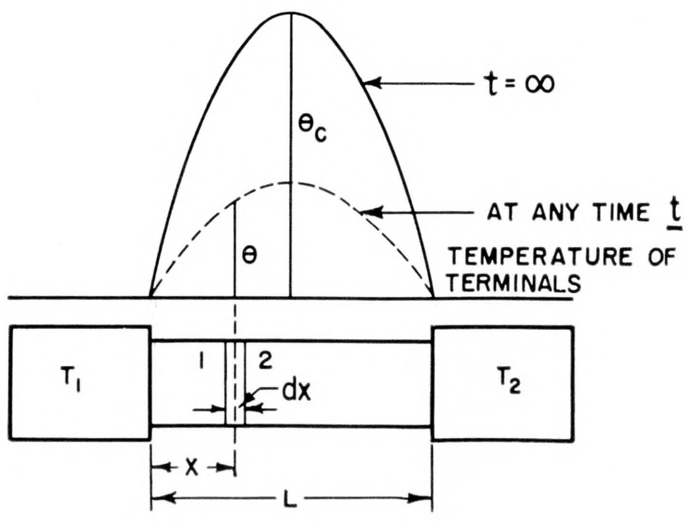

Figure 17—Temperature-time distribution diagram for a conductor connected between heat absorbing terminals, heated uniformly at a constant rate, in which the heat dissipated by convection is negligible relative to that conducted to the terminals.

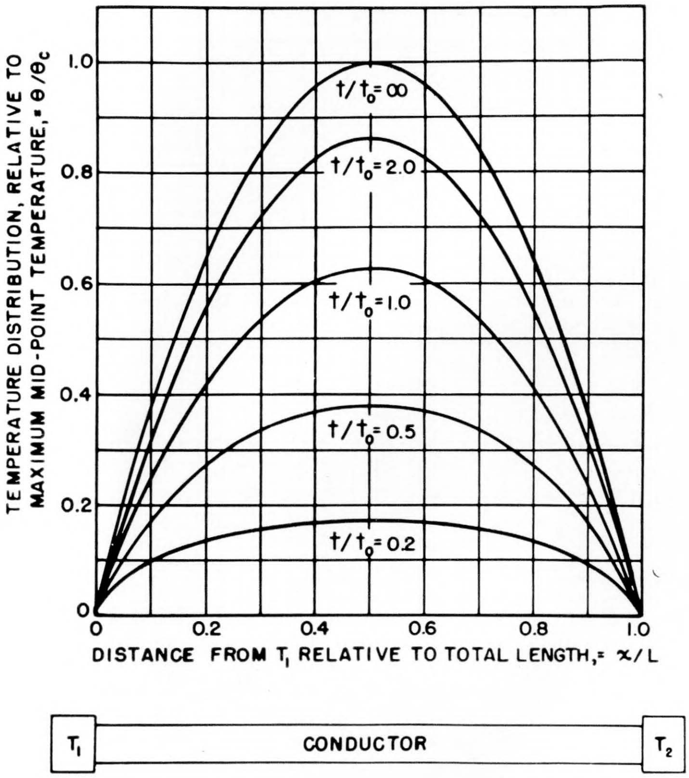

Figure 20—Distribution of temperature at various values of time, along a conductor connected between heat absorbing terminals, heated uniformly at a constant rate, and in which the heat dissipated by convection is negligible relative to that conducted to the terminals. Time is given relative to the time constant $t_o = (L^2 sm)/\pi^2 k$, and temperatures, relative to the final maximum temperature at the mid-point, $\theta_c = V^2/(8k\rho)$.

The equations for this condition can be deduced directly from the general Equation (111), by making the thermal length of the conductor, β, equal to zero. (See the first section of Part VII for a list of symbols used.)

22(a). Temperature Distribution at Any Time, in Terms of Final Temperature Distribution.

If β in Equation (111) is made zero, then, as will be shown later, the equation becomes

$$\frac{\theta}{\theta_c} = \frac{4(Lx - x^2)}{L^2} - \frac{32}{\pi^3}\left[\epsilon^{-\frac{t}{t_o}} \sin\frac{\pi x}{L} + \frac{\epsilon^{-\frac{3^2 t}{t_o}}}{3^3} \sin\frac{3\pi x}{L} + \frac{\epsilon^{-\frac{5^2 t}{t_o}}}{5^3} \sin\frac{5\pi x}{L} + \dots\right] \quad (121)$$

in which the principal time constant $t_o = L^2/(\pi^2 \alpha)$, and $\theta_c = V^2/8k\rho$ is the final temperature elevation at the mid-point of the conductor.

The final distribution of temperature is deduced from Equation (121) by making $t = \infty$ and we have the temperature at any point x,

$$\theta\Big]_{t=\infty} = \frac{4(Lx - x^2)\theta_c}{L^2} \quad (122)$$

This is the equation of a parabola between $x = 0$, and $x = L$ with its maximum value θ_c at the mid-point $x = L/2$. This is the same equation as given for the steady state distribution of temperature in the reference paper[(1)].

22(b). Temperature Distribution Along the Conductor at Any Time, in Terms of the Final Temperature Elevation, θ_c at the Mid-Point.

In Equation (121), for brevity, let

$$\frac{4(Lx - x^2)}{L^2} = y_1$$

Then it can be shown by Fourier Analysis that this equation for a parabola can be expressed as

$$y_1 = \frac{32\theta_c}{\pi^3}\left[\sin\frac{\pi x}{L} + \frac{1}{3^3}\sin\frac{3\pi x}{L} + \frac{1}{5^3}\sin\frac{5\pi x}{L} + \dots\right] \quad (123)$$

When this value for y_1 is substituted in Equation (121) and the terms collected, we have

$$\frac{\theta}{\theta_c} = \frac{32}{\pi^3}\left[\left(1 - \epsilon^{-\frac{t}{t_o}}\right)\sin\frac{\pi x}{L} + \frac{\left(1 - \epsilon^{-\frac{3^2 t}{t_o}}\right)}{3^3}\sin\frac{3\pi x}{L} + \frac{\left(1 - \epsilon^{-\frac{5^2 t}{t_o}}\right)}{5^3}\sin\frac{5\pi x}{L} + \dots\right] \quad (124)$$

Figure 20 is a graphical representation of this equation and shows the distribution of temperature at various values of time. To make the curve useful for general application, time is given in terms of the principal time constant t_o.

22(c). Temperature at the Mid-Point of the Conductor at Any Time.

In practice, the temperature at the mid-point of the conductor is usually the most important, as it is the maximum temperature. It is found by making $x = L/2$ in Equation (124) which after making the substitution, as will be shown later, becomes

$$\frac{\theta_{ct}}{\theta_c} = 1 - \frac{32}{\pi^3}\left[\epsilon^{-\frac{t}{t_o}} - \frac{\epsilon^{-\frac{3^2 t}{t_o}}}{3^3} + \frac{\epsilon^{-\frac{5^2 t}{t_o}}}{5^3} - \dots\right] \quad (125)$$

Where θ_{ct} is the temperature elevation at the mid-point of the conductor at any time t, and θ_c is the final mid-point temperature elevation.

Figure 21, computed by using Equation (125), shows the heating curve of the mid-point of a conductor connected between terminals, in which the heat loss by convection is negligible relative to that conducted to the terminals. The curves may be used to determine the mid-point temperature of any such conductor provided the time constant is known or can be computed.

22(d). Approximate Equation for Mid-Point Heating.

The series part of Equation (125) converges very rapidly. After a time equal to about 0.3 t_o has elapsed, the sum of the terms in the series beyond the first is less than 1.0 per cent of the first term. Therefore, for most practical purposes, Equation (125) may be used in the approximate form

$$\left.\frac{\theta_{ct}}{\theta_c}\right]_{t/t_o>0.3} \approx \left(1-\frac{32}{\pi^3}\epsilon^{-\frac{t}{t_o}}\right) \qquad (126)$$

22(e). Increase in Temperature at the Mid-Point of the Conductor During a Time Equal to the Time Constant.

In Equation (125) if the time t is made equal to the time constant t_o, that is, $t/t_o=1$ we find that

$$\left.\frac{\theta_{ct}}{\theta_c}\right]_{t=t_o} = 0.6203 \qquad (127)$$

which means that in a time equal to the time constant of the conductor, the temperature difference between the mid-point and a terminal will have reached about 62 per cent of its final value. It will be remembered that in the case of heating of simple bodies, the corresponding value was 63.2 per cent, which is not greatly different.

22(f). Rate of Increase in Temperature at the Mid-Point of the Conductor at Any Time.

By differentiating Equation (125) with respect to t/t_o we have

$$\frac{d\left(\frac{\theta_{ct}}{\theta_c}\right)}{d\left(\frac{t}{t_o}\right)} = \frac{32}{\pi^3}\left[\epsilon^{-\frac{t}{t_o}} - \frac{1}{3}\epsilon^{-\frac{3^2t}{t_o}} + \frac{1}{5}\epsilon^{-\frac{5^2t}{t_o}} - \ldots\right] \qquad (128)$$

which gives the rate of increase at any time t. The initial rate of increase, that is at $t/t_o=0$, is usually the most interesting. When t/t_o is made zero in Equation (128) we have

$$\frac{d\left(\frac{\theta_{ct}}{\theta_c}\right)}{d\left(\frac{t}{t_o}\right)} = \frac{32}{\pi^3}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\right) = \frac{32}{\pi^3}\times\frac{\pi}{4} = \frac{8}{\pi_2} \qquad (129)$$

since the sum of the series is $\pi/4$. That is, $8/\pi^2=0.81057$ is the initial rate at which the temperature elevation at the mid-point of the conductor increases with time, relative to the final temperature elevation θ_c, and the time constant t_o respectively.

The initial rate, therefore, is such that if it continued unchanged, the final temperature θ_c would be reached in a time equal to $(\pi^2)/8$ times the time constant, that is 1.2337 t_o.

In the case of the heating of simple bodies, it was found that under similar conditions, the final temperature would be reached in a time equal to the time constant, $t=t_o$.

22(g). Temperature of the Mid-Point of the Conductor Reached During Heavy Overloads for Short Periods.

In computing the temperature elevation of the conductor resulting from short time heavy overloads, it is assumed that the specific heat and conductivities are, for practical purposes, the same as at normal temperatures. It is also assumed that all of the heat not absorbed by the material is conducted to the terminals.

Let I = Overload Current and θ_c the Final Temperature Elevation at the mid-point for the overload current, if all material properties under normal conditions continued constant.

The temperature θ_c for this case is in general a fictitious value, since it may be beyond the danger point and even beyond the melting point of the material. It is merely a basis for determining the initial rate of increase in temperature. Then from the relation given under Equation (121) as proved earlier in this series

$$\theta_c = V^2/8k\rho \qquad (130)$$

Where V, k, and ρ, are the voltage between terminals for the current I, thermal conductivity, and resistivity of the conductor, respectively.

Then if the time constant t_o is known, the curves in Figure 21 can be used to determine the temperature elevation θ at any time t, as

$$\theta_c = (V^2/8k\rho)\times(\text{value of } \theta/\theta_c \text{ from curve}) \qquad (131)$$

If the temperature elevation for normal current is known, which is usually the case, then θ_c can be computed from the normal temperature by remembering

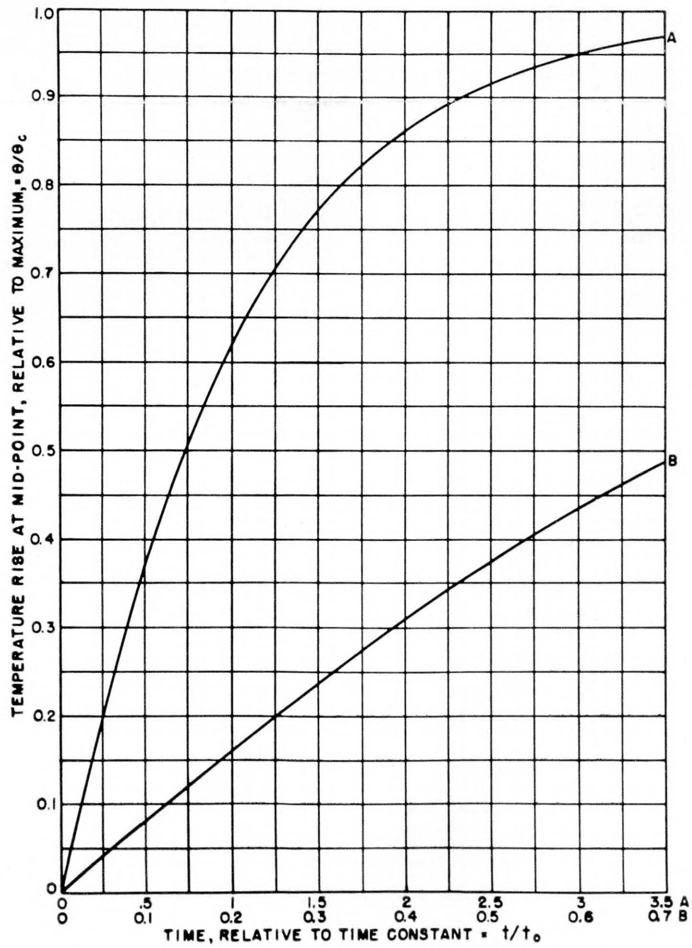

Figure 21—Temperature elevation at any time at the mid-point of a conductor connected between heat absorbing terminals, heated uniformly at a constant rate, and in which the heat dissipated by convection is negligible relative to that conducted to the terminals.

that the temperature elevation is proportional to the square of the current.

22(h). Example:

Assume a 50-ampere, 50-millivolt manganin shunt having a conductor 2 inches long. Find the time that a current of 400 amperes can be passed through it so that the temperature elevation shall reach but not exceed 250 deg. C. Then using Equation (130).

$$\theta_c = V^2/8k\rho$$

In which for manganin, $k = 0.26$ watt-cm.-deg. C, and $\rho = 40 \times 10^{-6}$ ohm-cm. cube from which $k\rho = 10.4 \times 10^{-6}$. $V = (400/50) \times 0.050 = 0.4$ volt.
Then:

$$\theta_c = (0.4)^2/(8 \times 10.4 \times 10^{-6}) = 1923 \text{ deg. C.}$$

which as referred to previously, cannot be realized physically, but is a mathematical basis for computing the initial rate of increase. Then the ratio of the desired temperature increase $\theta = 250$ *deg. C*, to the final increase $\theta_c = 1926$ is $\theta/\theta_c = 250/1923 = 0.130$. Then from the curve in Figure 21, the corresponding value of $t/t_o = 0.160$, that is, in a time equal to 0.160 times the time constant, the temperature elevation will have reached 250 deg. C. If the time constant is known, the actual time in seconds can be computed. If the time constant is not known, it can be computed readily from the value given under Equation (121) namely:

$$t_o = (L^2)/(\pi^2 \alpha)$$

where $\alpha = k/(sm)$

in which, for manganin, s = specific heat capacity = 0.42 joule per gram per deg. C; m = density = 8.5 grams per cu. cm.; and $k = 0.26$ watt-cm.-deg. C.
Then:

$$\alpha = 0.26/(0.42 \times 8.5) = 0.0728$$

$$L = \text{length} = 2 \times 2.54 = 5.08 \text{ cm.}$$

and

$$t_o = (5.08)^2/(\pi^2 \times 0.0728) = 35.8 \text{ seconds.}$$

From this the actual time to reach a temperature elevation of 250 deg. C is $t = 0.16\ t_o = 0.16 \times 35.8 = 5.73$ *sec.*

22(i). Derivation of Equations.

General Equation (111):
Referring to Figure 17. Let:

θ = temperature at x above that of terminal T_1.
A = area of cross-section of conductor; cm².
w = rate at which heat is applied per cm. length of conductor; watts/cm.
t = time after initial application of heat; seconds.
h = rate of heat convection loss per deg. C temperature elevation, per unit length of conductor; watts/cm./deg. C.

Let the positive direction be that corresponding to increasing values of x.

From physical reasoning, considering two sections δx apart, the heat which passes through section 2 in the positive direction in δt is equal to that passing through section 1 in the same direction and time, plus the heat generated in δx during δt, less the amount of heat absorbed by the material in δx resulting from a change in temperature $\delta\theta$ in δt, and less the heat lost by convection to the surrounding medium from δx during δt. Now, considering the positive direction:

Heat passing section 2 in $\delta t = -Ak(\delta\theta_2/\delta x)\,\delta t$.
Heat passing section 1 in $\delta t = -Ak(\delta\theta_1/\delta x)\,\delta t$.
Heat generated in δx during $\delta t = w\,\delta x\,\delta t$.
Heat absorbed in δx for $\delta\theta$ change in temperature $= Ams\,\delta x\,\delta\theta$.
Heat dissipated by convection in δx during $\delta t = h\theta\,\delta x\,\delta t$.

Then following the physical reasoning given above we have:

$$-Ak\frac{\delta\theta_2}{\delta x}\delta t = -Ak\frac{\delta\theta_1}{\delta x}\delta t + w\,\delta x\,\delta t - Ams\,\delta x\,\delta\theta - h\theta\,\delta x\,\delta t$$

Then

$$Ak\left(\frac{\delta\theta_1}{\delta x} - \frac{\delta\theta_2}{\delta x}\right)\delta t = w\,\delta x\,\delta t - Ams\,\delta x\,\delta\theta - h\theta\,\delta x\,\delta t \qquad (132)$$

But

$$\frac{\delta\theta_2}{\delta x} = \frac{\delta\theta_1}{\delta x} + \frac{\delta}{\delta x}\left(\frac{\delta\theta}{\delta x}\right)\delta x = \frac{\delta\theta_1}{\delta x} + \left(\frac{\delta^2\theta}{\delta x^2}\right)\delta x$$

Substitute this in Equation (132) and divide through by $Ak\,\delta x\,\delta t$, and we have

$$\frac{\delta^2\theta}{\delta x^2} = \frac{ms}{k}\left(\frac{\delta\theta}{\delta t}\right) - \frac{w}{Ak} + \frac{h\theta}{Ak} \qquad (133)$$

For brevity let $k/ms = \alpha$, $w/Ak = U$, and $h/Ak = u^2$
Then Equation (133) becomes

$$\frac{\delta^2\theta}{\delta x^2} = \frac{1}{\alpha}\left(\frac{\delta\theta}{\delta t}\right) + u^2\theta - U \qquad (134)$$

The solution of this differential equation follows the usual procedure for this form. Solutions similar to that given by the author may be found in the literature of heat conduction, but detailed steps are not given, nor of course are the application equations. As this series of articles is intended for engineering use, the solution of the equations is given in greater detail than would be required by the mathematician, so that especially the young engineer, by following the solution of a definite practical problem, may be able to apply the principles to other problems as they arise.

$$\text{To solve, let } \theta = z + F(x) \qquad (135)$$

Where $F(x)$ is a function of x only, and z is a function of both x and t. Then differentiate Equation (135) twice with respect to x, and we have

$$\frac{\delta^2\theta}{\delta x^2} = \frac{\delta^2 z}{\delta x^2} + F''(x)$$

Again, differentiate Equation (135) with respect to t and we have

$$\frac{\delta\theta}{\delta t} = \frac{\delta z}{\delta t}$$

Then substitute these two sets of values and the value of θ given in Equation (135), in Equation (134) and we have,

$$\frac{\delta^2 z}{\delta x^2}+F''(x)=\frac{1}{\alpha}\left(\frac{\delta z}{\delta t}\right)+u^2\left[z+F(x)\right]-U \qquad (136)$$

Now since $F(x)$ is arbitrary, let us select a value for it such that

$$F''(x)=u^2F(x)-U \qquad (137)$$

For brevity let $F(x)=y$

Then Equation (137) becomes

$$\frac{\delta^2 y}{\delta x^2}=-(U-u^2y) \qquad (138)$$

This equation has the same form as Equation (1) found by the writer in the reference paper[1] for the steady state condition, and its solution need not be repeated here. Applying to the general solution the boundary conditions pertaining to this present problem, namely, $\theta=O$, where $x=O$ and $x=L$, the final equation may be derived directly from Equation (5) of the reference paper[1] by remembering that the temperature elevation of the terminals, $T_1=T_2=O$, and that $U/u^2=w/h$. We then obtain

$$y=F(x)=\frac{U}{u^2}\left[1-\frac{sinh(L-x)u+sinh\ xu}{sinh\ Lu}\right] \qquad (139)$$

Substituting the value of $F''(x)$ from Equation (137) in (136), we obtain

$$\frac{\delta^2 z}{\delta x^2}=\frac{1}{\alpha}\left(\frac{\delta z}{\delta t}\right)+u^2z \qquad (140)$$

To integrate this equation let

$$z=XT \qquad (141)$$

where X is a function of x only, and T a function of t only. Then:

$$\frac{\delta z}{\delta x}=T\frac{\delta X}{\delta x},\ \frac{\delta^2 z}{\delta x^2}=T\frac{\delta^2 X}{\delta x^2},\ and\ \frac{\delta z}{\delta t}=X\frac{\delta T}{\delta t}.$$

which when substituted in Equation (140) results, after dividing through by XT, in

$$\frac{1}{X}\left(\frac{\delta^2 X}{\delta x^2}\right)=\frac{1}{\alpha T}\left(\frac{\delta T}{\delta t}\right)+u^2 \qquad (142)$$

The left-hand member of this equation is a function of x only, and the right-hand member a function of t only. Therefore the two, being equal, must be equal to a constant, say $-a^2$. The negative value of the square of the constant is used to simplify later work.

Then, $\frac{1}{X}\left(\frac{\delta^2 X}{\delta x^2}\right)=-a^2$ and $\frac{1}{\alpha T}\left(\frac{\delta T}{\delta t}\right)+u^2=-a^2$

From which

$$\frac{\delta^2 X}{\delta x^2}=-a^2X \qquad (143)$$

and

$$\frac{\delta T}{\delta t}=-\alpha T(a^2+u^2) \qquad (144)$$

Integrating (143) we have

either

$$x=\frac{1}{a}\ sin^{-1}\left(\frac{aX}{c}\right)+c_1 \qquad (145)$$

or

$$x=\frac{1}{a}\ cos^{-1}\left(\frac{aX}{c}\right)+c_2 \qquad (146)$$

where c, c_1, and c_2, are constants of integration. Rearranging Equations (145) and (146) we have

either

$$X=\frac{c}{a}\ sin\ (ax-c_1) \qquad (147)$$

or

$$X=\frac{c}{a}\ cos\ (ax-c_2) \qquad (148)$$

Since c_1a and c_2a are constants, let us designate them simply as c_1 and c_2.

Now, writing Equation (144) as $\delta t=-\left(\frac{1}{\alpha(a^2+u^2)}\right)\frac{\delta T}{T}$

and integrating, we have $\alpha(a^2+u^2)t=-log\ T+log\ c_3$

or

$$T=c_3\epsilon^{-\alpha(a^2+u^2)t} \qquad (149)$$

Substitute the values for T and X from Equations (147), (148) and (149), in Equation (141), and we have

either

$$z=c_3\epsilon^{-\alpha(a^2+u^2)t}\times\frac{c}{a}\ sin\ (ax-c_1) \qquad (150)$$

or

$$z=c_3\epsilon^{-\alpha(a^2+u^2)t}\times\frac{c}{a}\ cos\ (ax-c_2) \qquad (151)$$

Now, when $x=0$, $\theta=0$ by the conditions assumed; and also $F(x)=0$ when $x=0$ from Equation (139). Therefore from Equation (135) $z=0$, when $x=0$, and for this reason Equation (150) only, of the two Equations (150) and (151), will meet this condition, and then only if $c_1=0$.

Again z must be zero when $x=L$, since θ and $F(x)$ are zero when $x=L$, which is possible in Equation (150) if $a=(n\pi)/L$, where n is any integer. Then z becomes the sum of all the terms containing integers from 1 to ∞ that will be found possible in the problem. Then, remembering that $c_1=0$,

$$z=\sum c_3\epsilon^{-\alpha\left(\frac{n^2\pi^2}{L^2}+u^2\right)t}\times\frac{cL}{n\pi}\ sin\left(\frac{n\pi x}{L}\right) \qquad (152)$$

When $t=0$, then $\theta=0$, over the entire conductor. Then making $\theta=0$, in Equation (135), we have $F(x)=-z$ for $t=0$, and substituting this in Equation (152) after making $t=0$, we have

$$y=F(x)=-\sum\frac{c_3cL}{n\pi}\ sin\left(\frac{n\pi x}{L}\right) \qquad (153)$$

since c_3 and c are arbitrary, c_3c may be combined into a single constant, say c_4; then inserting the values for z and $F(x)$ in Equation (135) we obtain

$$\theta=\sum c_4\epsilon^{-\alpha\left(\frac{n^2\pi^2}{L^2}+u^2\right)t}\times\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)-\sum\frac{c_4L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) \quad (154)$$

Equation (139) for y can be expanded into the following Fourier Series;

$$y=\frac{U}{u^2}\sum A_n \sin\left(\frac{n\pi x}{L}\right) \quad (155)$$

where n is every odd integer from 1 to infinity, and A_n is the general Fourier coefficient. Performing the Fourier Analysis we find

$$A_n=\frac{4L^2u^2}{n\pi(L^2u^2+n^2\pi^2)},\ \text{(for odd value of } n\text{)} \quad (156)$$

Equating the values of y in Equations (153) and (155) we find

$$-\frac{c_4L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)=\frac{U}{u^2}A_n\sin\left(\frac{n\pi x}{L}\right)$$

and from this we can evaluate the constant c_4,

$$c_4=-\frac{UA_nn\pi}{Lu^2}$$

Then inserting the values for U under Equation (133) and A_n in Equation (156) we have

$$c_4=-\frac{4w}{Ak}\times\frac{L}{(L^2u^2+n^2\pi^2)}$$

Substituting this value for c_4 in Equation (154) we have

$$\theta=\frac{4wL^2}{Ak}\sum\frac{1-\epsilon^{-\alpha\left(\frac{n^2\pi^2}{L^2}+u^2\right)t}}{n\pi(n^2\pi^2+L^2u^2)}\sin\frac{n\pi x}{L} \quad (157)$$

which is one fundamental form of the equation desired. This can be written in terms of the final distribution of the conductor temperature, by substituting for the second term in Equation (154), which is y, its equivalent from Equation (139), then, after inserting the value of U, and designating the thermal length of the conductor $Lu=\beta$, we have

$$\theta=\frac{wL^2}{\beta^2Ak}\left(1-\frac{\sinh(1-x/L)\beta+\sinh x\beta/L}{\sinh\beta}\right)$$

$$-\frac{4wL^2}{Ak\pi}\sum\frac{\epsilon^{-\alpha\left(\frac{n^2\pi^2+\beta^2}{L^2}\right)t}}{n(n^2\pi^2+\beta^2)}\sin\frac{n\pi x}{L} \quad (158)$$

where n is every odd integer from 1 to infinity. This is Equation (111) which was to be proved.

Equation (121):

If β is made zero in the general Equation (158) as is necessary if no heat is lost by convection, we find that the first term, which is the final distribution of temperature, assumes the indeterminate form 0/0. This can be evaluated in the usual rigorous manner by differentiating numerator and denominator with respect to β. This is rather laborious. The following, however, is a simple but exact method of procedure.

Change the hyperbolic sines of the angles to their series equivalents, which for any angle q is $\sinh q=q+(q^3)/6+\ldots$. As q approaches zero, all terms beyond the second may be neglected. The second term must be included for the reason that without it the result becomes $1-1=0$, so that relative to zero, $(q^3)/6$ is not negligible. However, all terms beyond the second, when $q=0$, have no effect whatever upon the result. Then, after substituting the series equivalents for the hyperbolic sines in Equation (158), the first term of this equation, as β approaches zero, reduces to

$$\frac{wL^2}{Ak}\left(\frac{\frac{\beta^3x}{2L}-\frac{\beta^3x^2}{2L^2}}{\beta^3}\right)=4\theta_c\left(\frac{Lx-x^2}{L^2}\right)$$

since $wL^2/(2Ak)=4V^2/(8k\rho)=4\theta_c$

This is the steady state parabolic distribution of temperature along a conductor having no convection losses. If this is substituted for the first term in Equation (158) and β is made zero in the second term, then Equation (158) reduces to Equation (121) which was to be proved.

Equation (125):

When $x=L/2$ in Equation (124), we have

$$\left.\frac{\theta}{\theta_c}\right]_{x=\frac{L}{2}}=\frac{32}{\pi^3}\left[\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots\right)-\epsilon^{-\frac{t}{t_o}}+\frac{\epsilon^{-\frac{3^2t}{t_o}}}{3^3}-\ldots\right] \quad (159)$$

But the sum of the numerical series in the bracket is equal to $(\pi^3)/32$. Then Equation (159) becomes

$$\left.\frac{\theta}{\theta_c}\right]_{x=\frac{L}{2}}=1-\frac{32}{\pi^3}\left(\epsilon^{-\frac{t}{t_o}}-\frac{\epsilon^{-\frac{3^2t}{t_o}}}{3^3}+\frac{\epsilon^{-\frac{5^2t}{t_o}}}{5^3}-\ldots\right)$$

which is Equation (125).

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