The Shapley value and the interaction indices for bi-cooperative games

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February 19, 2003

1 Introduction

Cooperative game theory is based on the notion of game v defined on a set $N = \{1, \ldots, n\}$ of players.

Binary voting games are examples of such games [12] in which the value of v is limited to $\{-1, 1\}$. N is the set of voters, and for $S \subset N$, $v(S)$ is interpreted as the result of the vote ($+1$ means that the bill is accepted whereas -1 means that the bill is rejected) when S is the set of voters in favor, the other voters $N \setminus S$ being against. Voting games are typically decision rules in voting processes such as for bills.

Simple games are games in which v takes only the values 0 and 1. For a coalition S of players, $v(S) = 1$ if players of S win when they play together against the other ones, and $v(S) = 0$ otherwise (players of S lose).

For general games, $v(S)$ can take any value in $[0, 1]$ and represents the asset that all players of S will win if they play together against $N \setminus S$.

These concepts are not always sufficient to model correctly the reality. For instance, the binary voting games cannot represent decision rules in which *abstention* is an alternative option to the usual *yes* and *no* opinions. This leaded D. Felsenthal and M. Machover to introduce *ternary voting games* [3]. These voting games can be represented by a function v with two arguments, one for the *yes* voters and the other one for the *no* voters. Let

$$\mathcal{Q}(N) = \{(S, T) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid S \cap T = \emptyset\} \ .$$

The value of v is still limited to $\{-1, 1\}$, which means that a final decision shall be made. For $(S, T) \in \mathcal{Q}(N)$, $v(S, T)$ is interpreted as the result of the vote ($+1$ for acceptance and -1 for refusal) when S is the set of voters in favor, T is the set of voters that are against, the other voters $N \setminus (S \cup T)$ choosing abstention.

This concept of ternary voting game has been generalized by J.M. Bilbao *et al* in [1], yielding the definition of *bi-cooperative game*. A bi-cooperative game is a function $v : \mathcal{Q}(N) \to \mathbb{R}$ satisfying $v(\emptyset, \emptyset) = 0$. Let us denote by $\mathcal{G}^{[2]}(N)$ the set of all bi-cooperative games on N, and by $\mathcal{G}^{[2]}$ the set of all bi-cooperative games with finitely many players.

A bi-cooperative game will said to be *monotonic* if it is non-decreasing with respect to the first argument (i.e. $S \subset S' \Rightarrow v(S, T) \leq v(S', T)$) and non-increasing with respect to the second argument (i.e. $T \subset T' \Rightarrow v(S, T) \geq v(S, T')$). Finally, a bi-cooperative game will said to be *normalized* if v satisfies $v(N, \emptyset) = 1$ and $v(\emptyset, N) = -1$.

For $(S, T) \in \mathcal{Q}(N)$, $v(S, T)$ can be interpreted as the asset that all players of S will win or lose (depending on the sign of $v(S, T)$) if they play together against T, the players of $N \setminus (S \cup T)$ not taking part in the game. We will see later on other interpretations of $v(S, T)$. In the sequel, for $v(S, T)$, S is called the *defender* part whereas T is called the *defeater* part.

One of the main concerns in game theory is to define the notion of power or importance index of a player i, which is classically denoted by $\phi_i(v)$.

For binary voting games, the definition of the power index of a voter $i \in N$ is classically based on the notion of *binary roll-call* [2, 5]. A binary roll-call R is composed of an ordering σ_R of the voters and a coalition D_R which contains all voters that are in favor of the bill. The set of all roll-calls for the voters N is denoted by $\mathcal{B}_N$. Roll-calls are interpreted as follows. The voters are called in the order given by σ_R: $\sigma_R(1), \ldots, \sigma_R(n)$. When a voter i is called, he tells his opinion, that is to say *in favor* if $i \in D_R$ or *against* otherwise. Let j be the smallest index for which the result of the vote remains the same whatever voters called at position $j+1, \ldots, n$ say. It exists for any v and R. The *pivot* $Piv(v, R)$ for game v and roll-call R is the voter $\sigma_R(j)$ called at position j. Henceforth, the opinions of the voters called after $Piv(v, R)$ do not count. Moreover, $Piv(v, R)$ is the last voter whose opinion is really decisive. That $Piv(v, R)$ is decisive in the result of the vote is confirmed by the fact that if the bill is accepted (i.e. $v(D_R) = 1$) then $Piv(v, R)$ is necessarily in favor of the bill (i.e. $Piv(v, R) \in D_R$), whereas if the bill is rejected ($v(D_R) = -1$) then $Piv(v, R)$ is necessarily against of the bill (i.e. $Piv(v, R) \notin D_R$). So, it seems natural to define the power index $\phi_i(v)$ of voter i as the percentage of times i is the pivot in a binary roll-call

[4, 5] :

$$\phi_i(v) = \frac{|\{R \in \mathcal{B}_N \ , \ i = Piv(v,R)\}|}{|\mathcal{B}_N|} \ ,$$

where $|\mathcal{B}_N| = 2^n \, n!$. The importance index $\phi_i(v)$ is exactly the regular Shapley value [4].

For simple games, we consider, as previously, an ordering σ of the players. Let Γ be the set of all orderings of N. Starting from the empty coalition up to the full set N through coalitions of the form $\{\sigma(1), \ldots, \sigma(j)\}$, there exists one and only one index j such that $\{\sigma(1), \ldots, \sigma(j)\}$ loses whereas $\{\sigma(1), \ldots, \sigma(j+1)\}$ wins, i.e. $v(\{\sigma(1), \ldots, \sigma(j)\}) = 0$ and $v(\{\sigma(1), \ldots, \sigma(j+1)\}) = 1$. We say then that the player $I_\sigma := \sigma(j+1)$ *swings* coalition $\{\sigma(1), \ldots, \sigma(j)\}$. We will say also that i is a swing for σ and v. As in the case of binary voting games, player I_σ is decisive in the success of coalition $\{\sigma(1), \ldots, \sigma(j+1)\}$. The power index of a player i is now related to the number of swings of i when he joins coalitions [13] :

$$\phi_i(v) = \frac{|\{\sigma \in \Gamma \ , \ i = I_\sigma\}|}{|\Gamma|} \ ,$$

where $|\Gamma| = n!$. This is exactly the Shapley value.

For general games, the notion of swing is replaced by the difference of worth $v(S \cup i) - v(S)$ when player i joins coalition S. The Shapley value is then a mean value of these differences [13].

The first aim of this paper is to define the notion of importance index for bi-cooperative games. A definition of importance has already been proposed by D. Felsenthal and M. Machover for ternary voting games [3]. It extends in a natural way the notion of importance for binary voting games. First, ternary roll-calls are defined. A ternary roll-call R is composed of an ordering σ_R of the voters, a coalition D_R which contains all voters that are in favor of the bill and a coalition E_R which contains all voters that are against the bill. The set of all ternary roll-calls for the voters N is denoted by $\mathcal{T}_N$. As previously, when a voter i is called he tells his opinion, that is to say *in favor* if $i \in D_R$, *against* if $i \in E_R$ or *abstention* otherwise. The *pivot* $Piv(v,R)$ is defined as previously. The following definition is then proposed [3] :

$$\phi_i(v) = \frac{|\{R \in \mathcal{T}_N \ , \ i = Piv(v,R)\}|}{|\mathcal{T}_N|} \ ,$$

where $|\mathcal{T}_N| = 3^n \, n!$. This index will be referred to as *F-M power index.* As we will see in Section 2.2, for ternary voting games, if the bill is accepted (i.e. $v(D_R, E_R) = 1$) then $Piv(v, R)$ is necessarily either in favor of the bill or abstentionist, whereas if the bill is rejected ($v(D_R, E_R) = -1$) then $Piv(v, R)$ is necessarily either against the bill or abstentionist. As a consequence, we see that $Piv(v, R)$ is not always so decisive in the final decision made since it can be abstentionist. Because of that drawback, we do not stick to this definition of $\phi_i(v)$, and we are looking for other definitions of the importance index for bi-cooperative games.

Our starting point is the work of L.S. Weber [14]. The axiomatization of the Shapley index for cooperative games is based on four axioms: *linearity*, *dummy player*, *symmetry* and *efficiency*. These axioms are extended to the case of bi-cooperative games in section 2.1. Unlike the usual case, it turns out that they are not sufficient to uniquely determine the importance index. So, other axioms are necessary to obtain the expression of the importance index. The F-M power index is one particular example of such importance index. It is not easy to determine the last axioms that characterize this F-M power index in a natural way. We propose two axioms related to self-duality which depict some symmetry between defenders and defeaters. This leads to an expression of the importance index that is very close to the Shapley value.

In order to investigate more deeply the relationship between players, it is interesting to define also the notion of interaction index which quantifies the cooperation existing among players. The interaction index for games has been axiomatized by M. Grabisch and M. Roubens [9]. One single axiom is enough to define those indices from the importance indices: a *recursivity* axiom. This latter is naturally extended to the case of bi-cooperative games, leading to the expression of the interaction indices in this case (see Section 3).

Finally, we define the notion of bi-interaction indices which depicts the importance or interaction phenomena in some special situation (see section 4).

2 The importance index

The lattice $\mathcal{Q}(N)$ is endowed with the following order: $(S, T) \sqsubseteq (S', T')$ if $S \subset S'$ and $T \supset T'$. Upper letters will be used for subsets of players. For a coalition S, of players, the lowercase s will denote its coalition.

We consider the Shapley value as an operator on the set of bi-cooperative games $\phi : \mathcal{G}^{[2]}(N) \to \mathbb{R}^n$; $v \mapsto \phi^v$, for any finite support N, and coordinate i

of ϕ^v is denoted $\phi^v(i)$.

2.1 Generalization of the axioms of the usual Shapley value

We aim to define $\phi^v(i) \in \mathbb{R}$, which denotes the importance index of i with respect to bi-cooperative game v.

The first axiom that characterizes the Shapley value is *linearity* with respect to the game. This axiom states that if several games are combined linearly then the outcomes of each individual game shall be combined in the same way to obtain the outcome of the resulting game. This axiom is trivially extended to the case of bi-cooperative games.

Linearity (l): ϕ is linear on $\mathcal{G}^{[2]}(N)$.

Lemma 1 *$\{\phi^v(i)\}_{i\in N}$ satisfies* **(l)** *if and only if there exists $a^i_{S,T}$ for all $(S,T) \in \mathcal{Q}(N)$ such that*

$$\phi^v(i) = \sum_{(S,T)\in\mathcal{Q}(N)} a^i_{S,T} v(S,T) \ .$$

Proof : The *if* part of the proof is obvious and left to the reader.

Consider $\{\phi^v(i)\}_{i\in N}$ satisfying **(l)**. Let $U_{S,T}$ be defined by $U_{S,T}(S',T')$ equals 1 if $S = S'$ and $T = T'$, 0 otherwise. We have $v = \sum_{(S,T)\in\mathcal{Q}(N)} v(S,T) U_{S,T}$. By **(l)**,

$$\phi^v(i) = \sum_{(S,T)\in\mathcal{Q}(N)} v(S,T)\, \phi^{U_{S,T}}(i) \ .$$

Setting $a^i_{S,T} := \phi^{U_{S,T}}(i)$, we obtain the wished result. ■

The second axiom that characterizes the Shapley value is called dummy player or *null player*. It says that if a player i is null, i.e. $v(S\cup\{i\}) = v(S)$ for any $S \subset N \setminus \{i\}$, then this player does not contribute at all to any coalition and thus the importance index for this player vanishes. For bi-cooperative games, a player is said *null* if the asset is exactly the same if he joins the defenders or the defeaters.

Definition 1 *The player i is said to be* null *for the bi-cooperative game v if $v(S, T \cup \{i\}) = v(S,T) = v(S \cup \{i\}, T)$ for any $(S,T) \in \mathcal{Q}(N \setminus \{i\})$.*

We obtain the following axiom.

Null player (n): If a player i is null for the bi-cooperative game $v \in \mathcal{G}^{[2]}(N)$ then $\phi^v(i) = 0$.

Lemma 2 *$\{\phi^v(i)\}_{i\in N}$ satisfies* **(l)** *and* **(n)** *if and only if there exists $a^i_{S,T}$, $b^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N \setminus \{i\})$ such that*

$$\begin{aligned}\phi^v(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T}\left[v(S\cup\{i\},T) - v(S,T)\right] \\ &\quad + \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} b^i_{S,T}\left[v(S,T\cup\{i\}) - v(S,T)\right] \ .\end{aligned}$$

Proof : The *if* part of the proof is obvious and left to the reader.

Consider $\{\phi^v(i)\}_{i\in N}$ satisfying **(l)** and **(n)**. By Lemma 1, there exists $c^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N)$ such that

$$\phi^v(i) = \sum_{(S,T)\in\mathcal{Q}(N)} c^i_{S,T} v(S,T) \ .$$

We write

$$\begin{aligned}\phi^v(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} c^i_{S,T} v(S,T) + \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} c^i_{S\cup\{i\},T} v(S\cup\{i\},T) \\ &\quad + \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} c^i_{S,T\cup\{i\}} v(S,T\cup\{i\})\end{aligned}$$

Assume now that i is null for the bi-cooperative game v. Hence

$$\phi^v(i) = \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} v(S,T)\ \left[c^i_{S,T} + c^i_{S\cup\{i\},T} + c^i_{S,T\cup\{i\}}\right] \ .$$

This relation holds for any bi-cooperative game v such that i is null for v. Hence previous relation holds for any sub-game $v(S,T)$ with $(S,T) \in \mathcal{Q}(N \setminus \{i\})$. This gives for all $(S,T) \in \mathcal{Q}(N \setminus \{i\})$

$$c^i_{S,T} + c^i_{S\cup\{i\},T} + c^i_{S,T\cup\{i\}} = 0 \ .$$

Consequently, the expression of $\phi^v(i)$ can be rearranged in such a way to give the wished form. $c^i_{S\cup\{i\},T}$ is denoted by $a^i_{S,T}$ and $c^i_{S,T\cup\{i\}}$ by $b^i_{S,T}$. ■

The third axiom that characterizes the Shapley value is symmetry with respect to the players. It means that nothing special is done to one player compared to another one. In other words, the players are anonymous. This depicts fairness.

Let σ be a permutation defined on N. We define $\sigma(S) := \{\sigma(i), i \in S\}$ and $\sigma \circ v$ defined by $\sigma \circ v(\sigma(S), \sigma(T)) = v(S,T)$.

> **Symmetry (s)**: $\phi^{\sigma \circ v}(\sigma(i)) = \phi^v(i)$, for all $i \in N$ and for all $v \in \mathcal{G}^{[2]}(N)$.

Lemma 3 *$\{\phi^v(i)\}_{i\in N}$ satisfies* **(l)**, **(n)** *and* **(s)** *if and only if there exists $a^{(1)}_{s,t}$, $a^{(2)}_{s,t}$ for $(s,t) \in \{0,\dots,n-1\}^2$ with $s+t \le n-1$ such that*

$$\begin{aligned}\phi^v(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^{(1)}_{s,t}\left[v(S\cup\{i\},T) - v(S,T)\right] \\ &\quad + \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^{(2)}_{s,t}\left[v(S,T\cup\{i\}) - v(S,T)\right] .\end{aligned}$$

where $s = |S|$ and $t = |T|$.

Proof : The *if* part of the proof is obvious and left to the reader.

Let $\{\phi^v(i)\}_{i\in N}$ satisfy **(l)**, **(n)** and **(s)**. By Lemma 2, there exists $a^i_{S,T}$, $b^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N\setminus\{i\})$ such that

$$\begin{aligned}\phi^v(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T}\left[v(S\cup\{i\},T) - v(S,T)\right] \\ &\quad + \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} b^i_{S,T}\left[v(S,T\cup\{i\}) - v(S,T)\right] .\end{aligned}$$

We have

$$\begin{aligned}\phi^{\sigma\circ v}(\sigma(i)) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{\sigma(i)\})} a^{\sigma(i)}_{S,T}\left[\sigma\circ v(S\cup\{\sigma(i)\},T) - \sigma\circ v(S,T)\right] \\ &\quad + \sum_{(S,T)\in\mathcal{Q}(N\setminus\{\sigma(i)\})} b^{\sigma(i)}_{S,T}\left[\sigma\circ v(S,T\cup\{\sigma(i)\}) - \sigma\circ v(S,T)\right] .\end{aligned}$$

Set $S' = \sigma^{-1}(S)$ and $T' = \sigma^{-1}(T)$. We have $\sigma\circ v(\sigma(S')\cup\{\sigma(i)\},\sigma(T')) = \sigma\circ v(\sigma(S'\cup\{i\}),\sigma(T')) = v(S'\cup\{i\},T')$, $\sigma\circ v(\sigma(S'),\sigma(T')) = v(S',T')$ and $\sigma\circ v(\sigma(S'),\sigma(T')\cup\{\sigma(i)\}) = v(S',T'\cup\{i\})$. Hence

$$\begin{aligned}\phi^{\sigma\circ v}(\sigma(i)) &= \sum_{(S',T')\in\mathcal{Q}(N\setminus\{i\})} a^{\sigma(i)}_{\sigma(S'),\sigma(T')}\left[v(S'\cup\{i\},T') - v(S',T')\right] \\ &\quad + \sum_{(S',T')\in\mathcal{Q}(N\setminus\{i\})} b^{\sigma(i)}_{\sigma(S'),\sigma(T')}\left[v(S',T'\cup\{i\}) - v(S',T')\right]\end{aligned}$$

By **(s)**, we get

$$a^{\sigma(i)}_{\sigma(S),\sigma(T)} = a^i_{S,T} \quad , \quad b^{\sigma(i)}_{\sigma(S),\sigma(T)} = b^i_{S,T}$$

for any permutation σ, and any $(S,T) \in \mathcal{Q}(N \setminus \{i\})$. This shows that $a^i_{S,T}$ and $b^i_{S,T}$ do not depend on i, and depend only on the cardinality of S and T. $a^i_{S,T}$ is thus denoted by $a^{(1)}_{s,t}$ in short, and $b^i_{S,T}$ by $a^{(2)}_{s,t}$. ■

In regular games, the Shapley value is a distribution of the available overall resources $v(N)$. This leads to the final axiom that characterizes the Shapley value. It is called *efficiency* and reads $\sum_{i\in N} \phi^v(i) = v(N)$.

We are looking for the most natural extension of this axiom to the case of bi-cooperative games.

At first sight, it seems natural to keep exactly the same expression. Consider the case when $v(S,T)$ is interpreted as the asset that all players of S will win if they play together against T, the players of $N\setminus(S\cup T)$ not taking part in the game. It is reasonable in this case that the players share the result of the game which is $v(N,\emptyset)$. This leads to $\sum_{i\in N} \phi^v(i) = v(N,\emptyset)$.

As we shall see, other expressions are also possible. Consider indeed the following example. A set N of traders propose their services to two competitive companies A and B. The efficiency of these traders is measured regarding the profit company A obtains thanks to them. More precisely, let $v(S,T)$ be the difference between the profit of company A when S works for A and T works for B, and the profit of company A when none of N works for either A or B. From this definition, $v(\emptyset,\emptyset) = 0$. When no trader of N is working for company B, the traders of N working for A helps A to gain new customers originally faithful to company B, so that company A wins more money compared to when no trader of N works for A. This implies that $v(S,\emptyset) \geq 0$ for any S. On the contrary, when no trader of N is working for A, company B gets new customers whereas company A wins less money. Hence $v(\emptyset,T) \leq 0$ for any T. Reproducing previous arguments, it is easy to see that v is monotonic. It corresponds to a non-normalized bi-cooperative game. $v(N,\emptyset)$ is not necessarily equal to the opposite of $v(\emptyset,N)$.

Traders of N propose the head of company A to work for him. They then discuss about the wages of each individual trader of N. They explain that if they all work for B, then A looses $v(\emptyset,N)$. If they now work for A, A wins $v(N,\emptyset)$. This means that all traders of N brings an added worth of $v(N,\emptyset) - v(\emptyset,N)$ to company A. They therefore ask that their wages share this amount of money. The wage of trader i is denoted by $\phi^v(i)$. This gives $\sum_{i\in N} \phi^v(i) = v(N,\emptyset) - v(\emptyset,N)$. This is the second proposal for efficiency.

Let us see what is the best definition of efficiency. To this end, consider once more the case when $v(S,T)$ is interpreted as the asset that all players of S will win if they play together against T, the players of $N \setminus (S \cup T)$ not taking part in the game. The result of the game is null if the set of defenders is empty. More precisely, $v(\emptyset, T) = 0$ for any T. In particular, $v(\emptyset, N) = 0$. We thus obtain $\sum_{i \in N} \phi^v(i) = v(N, \emptyset) = v(N, \emptyset) - v(\emptyset, N)$. The two expressions of efficiency are exactly the same in this case.

Finally, consider the case of ternary voting games. For any ternary roll-call, there is one and only one pivot. This proves that $\sum_{i \in N} \phi_i(v) = 1$. It is easy to see in this case that this expression of efficiency is quite similar to the second one. Since $v(N, \emptyset) = 1$ and $v(\emptyset, N) = -1$, we write this formula as $\sum_{i \in N} \phi_i(v) = \frac{1}{2}(v(N, \emptyset) - v(\emptyset, N))$. Setting $\phi^v(i) = 2\,\phi_i(v)$, we obtain again the second proposal of efficiency. We see that the second proposal of efficiency is a more general expression. From above examples, the second expression encompasses the first as a particular case.

Efficiency (e): $\sum_{i \in N} \phi^v(i) = v(N, \emptyset) - v(\emptyset, N)$ for all $v \in \mathcal{G}^{[2]}(N)$.

Lemma 4 *$\{\phi^v(i)\}_{i \in N}$ satisfies* **(l)**, **(n)**, **(s)** *and* **(e)** *if and only if there exists $a^{(1)}_{s,t}$, $a^{(2)}_{s,t}$ for $(s,t) \in \{0, \ldots, n-1\}^2$ with $s + t \leq n - 1$ such that*

$$\begin{aligned} \phi^v(i) &= \sum_{(S,T) \in \mathcal{Q}(N \setminus \{i\})} a^{(1)}_{s,t} \left[v(S \cup \{i\}, T) - v(S,T) \right] \\ &\quad + \sum_{(S,T) \in \mathcal{Q}(N \setminus \{i\})} a^{(2)}_{s,t} \left[v(S, T \cup \{i\}) - v(S,T) \right] . \end{aligned}$$

where $s = |S|$ and $t = |T|$. Moreover

$$\begin{cases} n\, a^{(1)}_{n-1,0} = 1 \\ n\, a^{(2)}_{0,n-1} = -1 \\ \forall (s,t) \neq \{(n-1,0), (0,n-1)\} \\ \qquad s a^{(1)}_{s-1,t} + t a^{(2)}_{s,t-1} = (n-s-t)(a^{(1)}_{s,t} + a^{(2)}_{s,t}) \end{cases}$$

Proof : The *if* part of the proof is obvious and left to the reader.

Consider $\{\phi^v(i)\}_{i\in N}$ satisfying **(l)**, **(n)**, **(s)** and **(e)**. Then by Lemma 3, there exists $a^{(1)}_{s,t}$, $a^{(2)}_{s,t}$ for $(s,t) \in \{0,\dots,n-1\}^2$ with $s+t \leq n-1$ such that

$$\begin{aligned}\phi^v(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^{(1)}_{s,t}\left[v(S\cup\{i\},T) - v(S,T)\right] \\ &\quad + \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^{(2)}_{s,t}\left[v(S,T\cup\{i\}) - v(S,T)\right] \ .\end{aligned}$$

We write

$$\begin{aligned}\sum_{i\in N}\phi^v(i) &= \sum_{i\in N}\sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^{(1)}_{s,t} v(S\cup\{i\},T) \\ &\quad + \sum_{i\in N}\sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^{(2)}_{s,t} v(S,T\cup\{i\}) - \sum_{i\in N}\sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} (a^{(1)}_{s,t}+a^{(2)}_{s,t})v(S,T) \\ &= \sum_{(S',T')\in\mathcal{Q}(N)} v(S',T')\left[\sum_{i\in S'} a^{(1)}_{s'-1,t'} + \sum_{i\in T'} a^{(2)}_{s',t'-1} - \sum_{i\in N\setminus(S'\cup T')} (a^{(1)}_{s',t'}+a^{(2)}_{s',t'})\right] \\ &= \sum_{(S',T')\in\mathcal{Q}(N)} v(S',T')\left[s' a^{(1)}_{s'-1,t'} + t' a^{(2)}_{s',t'-1} - (n-s'-t')(a^{(1)}_{s',t'}+a^{(2)}_{s',t'})\right]\end{aligned}$$

Axiom **(e)** is thus satisfied if and only if

$$\begin{cases} n\, a^{(1)}_{n-1,0} = 1 \\ n\, a^{(2)}_{0,n-1} = -1 \\ \forall (s',t') \neq \{(n-1,0),(0,n-1)\} \\ \qquad s' a^{(1)}_{s'-1,t'} + t' a^{(2)}_{s',t'-1} = (n-s'-t')(a^{(1)}_{s',t'}+a^{(2)}_{s',t'}) \end{cases}$$

■

2.2 The Felsenthal-Machover importance index

Let us make the expression of the power index proposed by D. Felsenthal and M. Machover for ternary cooperative games [3] more explicit.

A ternary roll-call R is a triplet $R = (\sigma_R, D_R, E_R)$ composed of an ordering σ_R of the voters, a coalition D_R which contains all voters that are in favor of the bill, and a coalition E_R which contains all voters that are against the bill. The voters in $N\setminus(D_R\cup E_R)$ are abstentionist. The voters are called in the order $\sigma_R(1)$, $\sigma_R(2)$, …, $\sigma_R(n)$ given by σ_R. Let $L_k := \{\sigma_R(1),\dots,\sigma_R(k)\}$. When a voter i is called he tells his opinion, that is, *in favor* if $i \in D_R$, *against* if $i \in E_R$ or *abstention* otherwise. Let j be the smallest index for which the result of the vote remains the same whatever voters called at position $j+1, \ldots, n$ say. It exists for any v and R. The *pivot* $Piv(v,R)$ for game v and roll-call R is the voter $\sigma_R(j)$ called at position j. Henceforth, the opinions of the voters $N \setminus L_j$ called after $Piv(v,R)$ do not count. Then, by monotonicity of the bi-cooperative game v, the result of the vote remains the same whatever voters called at position $j+1, \ldots, n$ say if and only if $v(L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)) = v((L_j \cap D_R) \cup (N \setminus L_j), (L_j \cap E_R))$. As a consequence, the pivot $\sigma_R(j) = Piv(v,R)$ satisfies

$$\begin{cases} v(L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)) = v((L_j \cap D_R) \cup (N \setminus L_j), L_j \cap E_R) \\ v(L_{j-1} \cap D_R, (L_{j-1} \cap E_R) \cup (N \setminus L_{j-1})) \\ \qquad \neq v((L_{j-1} \cap D_R) \cup (N \setminus L_{j-1}), L_{j-1} \cap E_R) \end{cases} \tag{1}$$

Since

$$\begin{aligned} &(L_{j-1} \cap D_R, (L_{j-1} \cap E_R) \cup (N \setminus L_{j-1})) \sqsubseteq (L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)) \\ &\qquad \sqsubseteq ((L_j \cap D_R) \cup (N \setminus L_j), L_j \cap E_R) \\ &\qquad \sqsubseteq ((L_{j-1} \cap D_R) \cup (N \setminus L_{j-1}), L_{j-1} \cap E_R) \end{aligned}$$

we have

$$\begin{aligned} &v\left(L_{j-1} \cap D_R, (L_{j-1} \cap E_R) \cup (N \setminus L_{j-1})\right) \leq v\left(L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)\right) \\ &\qquad \leq v\left((L_j \cap D_R) \cup (N \setminus L_j), L_j \cap E_R\right) \\ &\qquad \leq v\left((L_{j-1} \cap D_R) \cup (N \setminus L_{j-1}), L_{j-1} \cap E_R\right) \ . \end{aligned}$$

Since the range of v is $\{-1, 1\}$ and due to (1), $\sigma_R(j)$ is the pivot $Piv(v,R)$ if and only if we are in one of the following two cases :

- $v(L_{j-1} \cap D_R, (L_{j-1} \cap E_R) \cup (N \setminus L_{j-1})) = -1$ and $v(L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)) = 1$.
- $v((L_j \cap D_R) \cup (N \setminus L_j), L_j \cap E_R) = -1$ and $v((L_{j-1} \cap D_R) \cup (N \setminus L_{j-1}), L_{j-1} \cap E_R) = 1$.

It is easy to see that the first case cannot happen when $\sigma_R(j) \in E_R$. If $\sigma_R(j) \in D_R$, the first case occurs if and only if $v(S \cup \{\sigma_R(j)\}, T) = 1$ and $v(S, T \cup \{\sigma_R(j)\}) = -1$ with $S = L_{j-1} \cap D_R$, $T = (L_{j-1} \cap E_R) \cup (N \setminus L_j)$, that is to say if and only if $v(S \cup \{\sigma_R(j)\}, T) - v(S, T \cup \{\sigma_R(j)\}) = 2$. If $\sigma_R(j) \notin D_R \cup E_R$, the first case occurs if and only if $v(S,T) = 1$ and $v(S, T \cup \{\sigma_R(j)\}) = -1$ with the same definition for S and T, that is to say if and only if $v(S,T) - v(S, T \cup \{\sigma_R(j)\}) = 2$. In this first case, the final decision is *yes* since $v(L_j \cap D_R, (L_j \cap E_R) \cup (N \setminus L_j)) = 1$. It is impossible that the *pivot* says *no* (i.e. $\sigma_R(j) \notin E_R$). Hence, if the bill is accepted then $Piv(v,R)$ is necessarily either in favor of the bill or abstentionist.

The second case cannot happen when $\sigma_R(j) \in D_R$. If $\sigma_R(j) \in E_R$, the second case occurs if and only if $v(S \cup \{\sigma_R(j)\}, T) = 1$ and $v(S, T \cup \{\sigma_R(j)\}) = -1$ with $S = (L_{j-1} \cap D_R) \cup (N \setminus L_j)$, $T = L_{j-1} \cap E_R$, that is to say if and only if $v(S \cup \{\sigma_R(j)\}, T) - v(S, T \cup \{\sigma_R(j)\}) = 2$. If $\sigma_R(j) \notin D_R \cup E_R$, the second case occurs if and only if $v(S \cup \{\sigma_R(j)\}, T) = 1$ and $v(S,T) = -1$ with the same definition for S and T, that is to say if and only if $v(S \cup \{\sigma_R(j)\}, T) - v(S,T) = 2$. In this second case, the final decision is *no* since $v((L_j \cap D_R) \cup (N \setminus L_j), L_j \cap E_R) = -1$. It is impossible that the *pivot* says *yes* (i.e. $\sigma_R(j) \notin D_R$). If the bill is rejected then $Piv(v,R)$ is necessarily either against the bill or abstentionist.

The F-M power index is defined as :

$$\phi^v_{\mathrm{FM}}(i) = \frac{|\{R \in \mathcal{T}_N \ , \ i = Piv(v,R)\}|}{|\mathcal{T}_N|} \ ,$$

where $|\mathcal{T}_N| = 3^n \, n!$.

From the previous cases, we obtain

$$\begin{aligned}
\phi^v_{\mathrm{FM}}(i) = \frac{1}{3^n \, n!} \sum_{(S,T) \in \mathcal{Q}(N \setminus \{i\})} \{ \\
\frac{v(S \cup \{i\}, T) - v(S, T \cup \{i\})}{2} \sum_{K \subset N \setminus \{i\}} k!(n-k-1)! \sum_{\substack{(D,E) \in \mathcal{Q}(N) \\ i \in D \ , \ S = D \cap K \\ T = (E \cap K) \cup (N \setminus (K \cup \{i\}))}} 1 \\
+ \frac{v(S,T) - v(S, T \cup \{i\})}{2} \sum_{K \subset N \setminus \{i\}} k!(n-k-1)! \sum_{\substack{(D,E) \in \mathcal{Q}(N) \\ i \in D \ , \ S = D \cap K \\ T = (E \cap K) \cup (N \setminus (K \cup \{i\}))}} 1 \\
+ \frac{v(S \cup \{i\}, T) - v(S, T \cup \{i\})}{2} \sum_{K \subset N \setminus \{i\}} k!(n-k-1)! \sum_{\substack{(D,E) \in \mathcal{Q}(N) \\ i \in E \ , \ T = E \cap K \\ S = (D \cap K) \cup (N \setminus (K \cup \{i\}))}} 1 \\
+ \frac{v(S \cup \{i\}, T) - v(S,T)}{2} \sum_{K \subset N \setminus \{i\}} k!(n-k-1)! \sum_{\substack{(D,E) \in \mathcal{Q}(N) \\ i \notin D \cup E \ , \ T = E \cap K \\ S = (D \cap K) \cup (N \setminus (K \cup \{i\}))}} 1 \}
\end{aligned}$$

This gives

$$\phi^v_{\rm FM}(i) = \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} \Big[\Big(\eta_s + \frac{\eta_t}{2}\Big)\left(v(S\cup\{i\},T) - v(S,T)\right) - \Big(\eta_t + \frac{\eta_s}{2}\Big)\left(v(S,T\cup\{i\}) - v(S,T)\right)\Big] \; .$$

where

$$\eta_s = \frac{1}{3^n n!}\sum_{m=0}^{s}\frac{m!(s-m)!}{s!}(n-1-s+m)!(s-m)!3^{s-m} \; .$$

From the expression of ϕ_i, the power index $\phi^v_{\rm FM}(i)$ satisfies the axioms **(l)**, **(n)**, **(s)**. Moreover, since for any ternary roll-call R there is one and only one pivot, one clearly has

$$\sum_{i\in N}\phi^v_{\rm FM}(i) = 1 \; .$$

Henceforth, since $v(N,\emptyset) = 1$ and $v(\emptyset, N) = -1$, $2\phi^v_{\rm FM}(i)$ satisfies the axioms **(l)**, **(n)**, **(s)** and **(e)**.

From the study of the two cases explained earlier, we see that $Piv(v,R)$ is not always so decisive in the final decision made since it can be abstentionist. An alternative definition of the power index would be to consider only the cases for which the pivot is not abstentionist, and thus have exactly the same opinion as the final result. From previous calculation, this amounts to remove the terms of the form $v(S\cup\{i\},T) - v(S,T)$ and $v(S,T) - v(S,T\cup\{i\})$ in the expression of $\phi^v_{\rm FM}(i)$. Consequently, $\phi^v_{\rm FM}(i)$ would contain only terms of the kind $v(S\cup\{i\},T) - v(S,T\cup\{i\})$.

2.3 Other proposal

One classical property satisfied by the Shapley value is self-duality [12, 13]. The dual v^* of a game v is defined by $v^*(S) = v(N) - v(N\setminus S)$. Self-duality means that $\phi_i(v^*) = \phi_i(v)$. Let us give the reason for simple games. Consider a permutation σ. Its dual σ^* is defined by $\sigma^*(k) = \sigma(n-k+1)$. It is easy to see that i is a swing for σ^* and v^* if and only if it is also a swing for σ and v. It is therefore natural that $\phi_i(v^*) = \phi_i(v)$.

For bi-cooperative games, the dual v^* of v can be defined by $v^*(S,T) = -v(T,S)$ for all $(S,T)\in\mathcal{Q}(N)$. The defenders and the defeaters are switched, and the abstentionists are untouched. The minus sign comes from the fact that if v is monotonic so is v^*. Clearly, $v^*\in\mathcal{G}^{[2]}(N)$ if $v\in\mathcal{G}^{[2]}(N)$. This definition of dual bi-cooperative games coincides with that proposed in [3] for ternary voting games. The authors noted also in [3] that their F-M power index is self-dual. So, it is reasonable to ask that ϕ satisfies also this property.

Self-dual (sd): $\phi^{v^*} = \phi^v$ for any v in $\mathcal{G}^{[2]}(N)$.

Lemma 5 *$\{\phi^v(i)\}_{i\in N}$ satisfies* **(l)**, **(n)** *and* **(sd)** *if and only if there exists $a^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N \setminus \{i\})$ such that*

$$\begin{aligned}\phi^v(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T}\left[v(S\cup\{i\},T) - v(S,T)\right] \\ &\quad - \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{T,S}\left[v(S,T\cup\{i\}) - v(S,T)\right] .\end{aligned}$$

Proof : We only prove the *only if* part of the proof. Consider $\{\phi^v(i)\}_{i\in N}$ satisfying **(l)**, **(n)** and **(sd)**. By Lemma 2, there exists $a^i_{S,T}$, $b^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N \setminus \{i\})$ such that

$$\begin{aligned}\phi^v(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T}\left[v(S\cup\{i\},T) - v(S,T)\right] \\ &\quad + \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} b^i_{S,T}\left[v(S,T\cup\{i\}) - v(S,T)\right] .\end{aligned}$$

We have

$$\begin{aligned}\phi^{v^*}(i) &= -\sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T}\left[v(T,S\cup\{i\}) - v(T,S)\right] \\ &\quad - \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} b^i_{S,T}\left[v(T\cup\{i\},S) - v(T,S)\right] .\end{aligned}$$

Hence by **(sd)**,

$$a^i_{S,T} = -b^i_{T,S} \quad \forall (S,T) \in \mathcal{Q}(N \setminus \{i\}) .$$

which concludes the proof. ■

For a game v, define v^*_i by $v^*_i(S) = v(\Pi_i(S))$ where $\Pi_i(S) = (S \cap \{i\}) \cup (N \setminus (S \setminus \{i\}))$ for any S. The defenders except i are switched with the defeaters except i. Moreover, if v is monotone, then v^*_i is not monotone, but it is monotonic with respect to i in the sense that $v^*_i(S \cup \{i\}) \geq v^*_i(S)$ for all

$S \subset N \setminus \{i\}$. It is easy to see that $\phi_i(v_i^*) = \phi_i(v)$ for the usual Shapley value. This property basically comes from the fact that player i is unaffected in the transformation from v to v_i^*.

This property can be carried over to the case of bi-cooperative games. To this end, define $\Pi_i(S,T) = ((S \cap \{i\}) \cup (T \setminus \{i\}), (S \setminus \{i\}) \cup (T \cap \{i\}))$ and v_i^* by $v_i^*(S,T) = v(\Pi_i(S,T))$ for all $(S,T) \in \mathcal{Q}(N)$. All players except i are switched from the defender part to the defeater part, i being left untouched. If $v \in \mathcal{G}^{[2]}(N)$ then $v_i^* \in \mathcal{G}^{[2]}(N)$. Moreover, if v is monotone, then v_i^* is monotonic with respect to i in the sense that $v_i^*(S \cup \{i\}, T) \geq v_i^*(S,T) \geq v_i^*(S, T \cup \{i\})$ for all $(S,T) \in \mathcal{Q}(N \setminus \{i\})$. The transformation from v to v_i^* does not affect the importance index of player i, as for regular cooperative games.

Mirror (m): For any $v \in \mathcal{G}^{[2]}(N)$ and any $i \in N$, $\phi^{v_i^*}(i) = \phi^v(i)$.

Lemma 6 *$\{\phi^v(i)\}_{i \in N}$ satisfies* **(l)**, **(n)**, **(sd)** *and* **(m)** *if and only if there exists $a^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N \setminus \{i\})$ such that*

$$\phi^v(i) = \sum_{(S,T) \in \mathcal{Q}(N \setminus \{i\})} a^i_{S,T} \left[v(S \cup \{i\}, T) - v(S, T \cup \{i\}) \right] \ .$$

Compared to the expression of $\phi^v(i)$ given in Lemma 2, there are no more terms of the form $v(S \cup \{i\}, T) - v(S,T)$ and $v(S,T) - v(S, T \cup \{i\})$. More precisely, $\phi^v(i)$ depends only the terms $v(S \cup \{i\}, T) - v(S, T \cup \{i\})$. As we have seen before, this corresponds to remove the abstentionist pivots.

The F-M power index does not satisfy axiom **(m)**.

Proof : We prove only the *only if* part. Consider $\{\phi^v(i)\}_{i \in N}$ satisfying **(l)**, **(n)**, **(sd)** and **(m)**. By Lemma 5, there exists $a^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N \setminus \{i\})$ such that

$$\begin{aligned} \phi^v(i) &= \sum_{(S,T) \in \mathcal{Q}(N \setminus \{i\})} a^i_{S,T} \left[v(S \cup \{i\}, T) - v(S,T) \right] \\ &\quad - \sum_{(S,T) \in \mathcal{Q}(N \setminus \{i\})} a^i_{T,S} \left[v(S, T \cup \{i\}) - v(S,T) \right] \ . \end{aligned}$$

We have

$$\begin{aligned} \phi^{v_i^*}(i) &= \sum_{(S,T) \in \mathcal{Q}(N \setminus \{i\})} a^i_{S,T} \left[v(T \cup \{i\}, S) - v(T,S) \right] \\ &\quad - \sum_{(S,T) \in \mathcal{Q}(N \setminus \{i\})} a^i_{T,S} \left[v(T, S \cup \{i\}) - v(T,S) \right] \ . \end{aligned}$$

By **(m)**, we obtain for all $(S,T) \in \mathcal{Q}(N \setminus \{i\})$

$$a^i_{S,T} = a^i_{T,S} \ .$$

■

There are alternative ways to remove the abstentionists in the expression of $\phi^v(i)$. Axioms **(sd)** and **(m)** can be replaced the following one **(np)**. For $i \in N$, consider two games for which the result is exactly the same when i belongs either to the defeater or the defender part. These two games differs when i is abstentionist. The case when i is abstentionist shall not be considered in the computation of the importance index so that these two games shall have the same importance for player i.

> **Negative-positive (np)**: Let $i \in N$, and v_1, v_2 in $\mathcal{G}^{[2]}(N)$ such that for any $(A,B) \in \mathcal{Q}(N \setminus \{i\})$
>
> $$v_1(A\cup\{i\}, B) = v_2(A\cup\{i\}, B) \ , \ v_1(A, B\cup\{i\}) = v_2(A, B\cup\{i\}) \ ,$$
>
> and
>
> $$v_1(A,B) = v_1(A \cup \{i\}, B) \ , \ v_2(A,B) = v_2(A, B \cup \{i\}) \ .$$
>
> Then $\phi^{v_1}(i) = \phi^{v_2}(i)$.

Axiom **(np)** states that the importance index does not depend on the result of the game when i is abstentionist. Replacing **(sd)** and **(m)** by **(np)**, we obtain the same result as in Lemma 6.

Lemma 7 *$\{\phi^v(i)\}_{i\in N}$ satisfies* **(l)**, **(n)** *and* **(np)** *if and only if there exists $a^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N \setminus \{i\})$ such that*

$$\phi^v(i) = \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T} \left[v(S \cup \{i\}, T) - v(S, T \cup \{i\})\right] \ .$$

Proof : The *if* part of the proof is obvious and left to the reader.

Consider $\{\phi^v(i)\}_{i\in N}$ satisfying **(l)**, **(n)** and **(np)**. By Lemma 2, there exists $a^i_{S,T}$ for $(S,T) \in \mathcal{Q}(N)$ with $\{i\} \in S,T$ such that

$$\begin{aligned}\phi^v(i) \ &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S\cup\{i\},T} \left[v(S \cup \{i\}, T) - v(S,T)\right] \\ &+ \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T\cup\{i\}} \left[v(S, T \cup \{i\}) - v(S,T)\right] \ .\end{aligned}$$

We have

$$\begin{aligned}\phi^{v_1}(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T\cup\{i\}}\left[v(S,T\cup\{i\})-v(S,T)\right] \\ &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T\cup\{i\}}\left[v(S\cup\{i\},T)-v(S,T\cup\{i\})\right] \ ,\end{aligned}$$

and

$$\begin{aligned}\phi^{v_2}(i) &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S\cup\{i\},T}\left[v(S\cup\{i\},T)-v(S,T)\right] \\ &= \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S\cup\{i\},T}\left[v(S\cup\{i\},T)-v(S,T\cup\{i\})\right] \ .\end{aligned}$$

By **(np)**, we obtain $a^i_{S,T\cup\{i\}} = a^i_{S\cup\{i\},T}$ for any $(S,T)\in\mathcal{Q}(N\setminus\{i\})$, which we write $a^i_{S,T}$ in short. This gives the final result. ■

Lemma 8 *$\{\phi^v(i)\}_{i\in N}$ satisfies* **(l)**, **(n)**, **(sd)**, **(m)** *and* **(s)** *if and only if there exists $a_{s,t}$ for $(s,t)\in\{0,\dots,n-1\}^2$ with $s+t\le n-1$ such that*

$$\phi^v(i) = \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a_{s,t}\left[v(S\cup\{i\},T)-v(S,T\cup\{i\})\right] \ ,$$

where $s=|S|$ and $t=|T|$.

Proof : The *if* part of the proof is obvious and left to the reader.

Let $\{\phi^v(i)\}_{i\in N}$ satisfy **(l)**, **(n)**, **(sd)**, **(m)** and **(s)**. By lemma 6, there exists $a^i_{S,T}$ for $(S,T)\in\mathcal{Q}(N\setminus\{i\})$ such that

$$\phi^v(i) = \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a^i_{S,T}\left[v(S\cup\{i\},T)-v(S,T\cup\{i\})\right] \ .$$

By the proof of Lemma 3, the symmetry axiom **(s)** implies that the coefficient $a^i_{S,T}$ depends only on the cardinality of S and T. It is denoted by $a_{s,t}$ in short. ■

Lemma 9 *$\{\phi^v(i)\}_{i\in N}$ satisfies* **(l)**, **(n)**, **(sd)**, **(m)**, **(s)** *and* **(e)** *if and only if*

$$\phi^v(i) = \sum_{S\subset N\setminus\{i\}} \frac{s!\,(n-s-1)!}{n!}\left[v(S\cup\{i\},N\setminus(S\cup\{i\}))-v(S,N\setminus S)\right] \ .$$

Proof : The *if* part of the proof is obvious and left to the reader.

Consider $\{\phi^v(i)\}_{i\in N}$ satisfying **(l)**, **(n)**, **(sd)**, **(m)**, **(s)** and **(e)**. Then by Lemma 3, there exists $a_{s,t}$ for $(s,t) \in \{0,\dots,n-1\}^2$ with $s+t \le n-1$ such that

$$\phi^v(i) = \sum_{(S,T)\in\mathcal{Q}(N\setminus\{i\})} a_{s,t}\left[v(S\cup\{i\},T) - v(S,T\cup\{i\})\right] \ ,$$

where $s = |S|$ and $t = |T|$. Hence we have $a^{(1)}_{s,t} = a^{(2)}_{s,t} = a_{s,t}$. By Lemma 4, we obtain

$$\begin{cases} a_{n-1,0} = a_{0,n-1} = \frac{1}{n} \\ a_{s,0} = a_{0,s} = 0 \quad \forall s \in \{1,\dots,n-2\} \\ s\, a_{s-1,t} = t\, a_{s,t-1} \quad \forall s,t \in \{1,\dots,n-1\} \end{cases}$$

For any $(s,t) \in \{0,\dots,n-1\}^2$ with $s+t \le n-1$, we have

$$a_{s,t} = \frac{t}{s+1}\, a_{s+1,t-1} = \cdots = \frac{t!}{(s+1)(s+2)\cdots(s+t)}\, a_{s+t,0} \ .$$

Hence, when $s+t = n-1$ one obtains

$$a_{s,t} = \frac{s!\, t!}{(n-1)!}\, a_{n-1,0} = \frac{s!\, t!}{n!} \ .$$

When $s+t < n-1$, we get

$$a_{s,t} = 0 \ .$$

We conclude that

$$a_{s,t} \neq 0 \iff s+t = n-1 \iff S\cup T = N\setminus\{i\}$$

Hence

$$\phi^v(i) = \sum_{S\subset N\setminus\{i\}} \frac{s!\,(n-s-1)!}{n!}\left[v(S\cup\{i\}, N\setminus(S\cup\{i\})) - v(S, N\setminus S)\right] \ .$$

■

Let us end up this section by giving an interpretation of this index in the framework of ternary voting games. There is a group of voters all willing to vote for some bill. They are called in a given order σ, and they tell their opinion when called. When all individuals called at positions $1,\dots,j$ have voted, we know for sure at this point that the bill is accepted if the result of the vote does not depend on the opinions of the remaining voters. In the worst case, all the remaining voters could say *'no'*. So, when the first j voters have said *'yes'*, the bill is surely accepted if $v(\{\sigma(1),\ldots,\sigma(j)\},\{\sigma(j+1),\ldots,\sigma(n)\}) = 1$. We consider the first index j such that this property holds. We have

$$\begin{cases} v\left(\{\sigma(1),\ldots,\sigma(j-1)\},\{\sigma(j),\ldots,\sigma(n)\}\right) = -1 \ , \\ v\left(\{\sigma(1),\ldots,\sigma(j)\},\{\sigma(j+1),\ldots,\sigma(n)\}\right) = 1 \ . \end{cases}$$

In this case, we see that voter $\sigma(j)$ is decisive in the acceptance of the bill. We note $\sigma(j) =: I_\sigma$. So, a natural definition of the power index of a voter i consists then to look at the number of times i is decisive for all orderings σ :

$$\frac{|\{\sigma \in \Gamma\ ,\ i = I_\sigma\}|}{|\Gamma|} \ .$$

It is easy to see that this expression is exactly $\frac{1}{2}\phi^v(i)$. As a consequence, the *abstention* option is not considered in the expression of $\phi^v(i)$. The interpretation of $\phi^v(i)$ is quite close to that of the Shapley indices for voting games. Indeed, for the Shapley indices, the decisive player is the first one for which all individuals called before and up to him form a winning coalition. This means that the opinions of the remaining voters do not count.

3 The interaction indices

From now on, $\phi^n(i)$ is also denoted by $I^v(\{i\})$.

In the case of usual games, the interaction index is constructed from the Shapley index using a recursivity axiom [9]. It is based on the notions of restricted game and reduced game.

We first define restricted and reduced bi-cooperative games. A restricted game is a game in the absence of the players of a coalition K. It is defined for bi-cooperative games in the same way :

$$v^{N\setminus K} : \begin{array}{rcl} \mathcal{Q}(N\setminus K) & \to & \mathbb{R} \\ (S,T) & \mapsto & v(S,T) \end{array}$$

If $v \in \mathcal{G}^{[2]}(N)$, then $v^{N\setminus K} \in \mathcal{G}^{[2]}(N\setminus K)$.

A reduced game is a game in which the players of a coalition K are either playing together or not playing. They are never considered individually. Consequently, they can be identified as a single player denoted by $[K]$. The set of players becomes $N_{[K]} := (N\setminus K)\cup\{[K]\}$. Define the mapping

$$\rho_{[K]} : \begin{array}{rcl} N_{[K]} & \to & N \\ S & \mapsto & \begin{cases} S \text{ if } [K] \notin S \\ (S\setminus\{[K]\})\cup K \text{ otherwise} \end{cases} \end{array}$$

Then the reduced game $v^{[K]}$ is defined by

$$v^{[K]}: \begin{array}{rcl} \mathcal{Q}(N_{[K]}) & \to & \mathbb{R} \\ (S,T) & \mapsto & v\left(\rho_{[K]}(S), \rho_{[K]}(T)\right) \end{array}$$

If $v \in \mathcal{G}^{[2]}(N)$, then $v^{N\setminus K} \in \mathcal{G}^{[2]}(N_{[K]})$.

As in [9], the interaction index shall satisfy the following recursivity property.

Recursivity (R): For any $v \in \mathcal{G}^{[2]}$, for any $\emptyset \neq S \subset N$,

$$I^v(S) = I^{v^{[S]}}([S]) - \sum_{K\subset S\,,\,K\neq\emptyset,S} I^{v^{N\setminus K}}(S \setminus K)\ .$$

Let us introduce some notation. Set

$$\Delta_{\{i\},\emptyset} v(S,T) = v(S \cup \{i\}\,,T) - v(S,T) \quad , \quad \Delta_{\emptyset,\{i\}} v(S,T) = v(S, T\cup\{i\}) - v(S,T)\ ,$$

and

$$\Delta_{A,B} v(S,T) := \sum_{K\subset A, M\subset B} (-1)^{(a-k)+(b-m)} v(S\cup K, T\cup M)\ .$$

We clearly have

$$\Delta_{A,B} v(S,T) = \left(\circ_{i\in A}\Delta_{\{i\},\emptyset}\right) \circ \left(\circ_{i\in B}\Delta_{\emptyset,\{i\}}\right) v(S,T) \qquad (2)$$

All the $\circ$ are commutative.

Lemma 10 *$\{I^v(S)\}_{S\in\mathcal{P}(N)}$ satisfies* **(R)** *if and only if*

$$I^v(S) = \sum_{T\subset N\setminus S} \frac{t!\,(n-s-t)!}{(n-s+1)!} \left[\Delta_{S,\emptyset} v\,(T, N\setminus (T\cup S)) - \Delta_{\emptyset,S} v\,(T, N\setminus(T\cup S))\right]\ .$$

Proof : We prove it by induction of $|S|$. For $|S| = 1$, this corresponds to the Shapley value. Assume it holds up to $s-1$. We rewrite it as $\forall S' \subset N$ with $|S'| < s$

$$I^v(S') = \sum_{T\subset N\setminus S'} \frac{t!\,(n-s'-t)!}{(n-s'+1)!} \left\{ \sum_{L\subset S'\,,\,L\neq\emptyset} (-1)^{s'-l} v\,(L\cup T, N\setminus(T\cup S')) - \sum_{L\subset S'\,,\,L\neq\emptyset} (-1)^{s'-l} v\,(T, (N\setminus(T\cup S'))\cup L) \right\}\ .$$

We have by the induction assumption

$$I^{v^{[S]}}([S]) = \sum_{T\subset N\setminus S} \frac{t!\,(n-s-t)!}{(n-s+1)!}\left[v(S\cup T, N\setminus(S\cup T)) - v(T, N\setminus T)\right] \ .$$

$$\begin{aligned} I^{v^{N\setminus K}}(S\setminus K) \quad = \quad & \sum_{T\subset N\setminus S} \frac{t!\,(n-s-t)!}{(n-s+1)!}\left\{ \sum_{L\subset S\setminus K\,,\,L\neq\emptyset} (-1)^{s-k-l} v\left(L\cup T, N\setminus(T\cup S)\right) \right. \\ & \left. - \sum_{L\subset S\setminus K\,,\,L\neq\emptyset} (-1)^{s-k-l} v\left(T, (N\setminus(T\cup S))\cup L\right) \right\} \ . \end{aligned}$$

Hence

$$\begin{aligned} I^v(S) = & \sum_{T\subset N\setminus S} \frac{t!\,(n-s-t)!}{(n-s+1)!} \\ & \left\{ \left[v(S\cup T, N\setminus(S\cup T)) - \sum_{K\subset S\,,\,K\neq\emptyset,S}\ \sum_{L\subset S\setminus K\,,\,L\neq\emptyset} (-1)^{s-k-l} v\left(L\cup T, N\setminus(T\cup S)\right) \right] \right. \\ & \left. \left[v(T, N\setminus T) - \sum_{K\subset S\,,\,K\neq\emptyset,S}\ \sum_{L\subset S\setminus K\,,\,L\neq\emptyset} (-1)^{s-k-l} v\left(T, (N\setminus(T\cup S))\cup L\right) \right] \right\} \\ = \quad & \sum_{T\subset N\setminus S} \frac{t!\,(n-s-t)!}{(n-s+1)!} \\ & \left\{ \left[v(S\cup T, N\setminus(S\cup T)) - \sum_{L\subset S\,,\,L\neq\emptyset,S} v\left(L\cup T, N\setminus(T\cup S)\right) \times \sum_{K\subset S\setminus L\,,\,K\neq\emptyset} (-1)^{s-k-l} \right] \right. \\ & \left. - \left[v(T, N\setminus T) - \sum_{L\subset S\,,\,L\neq\emptyset,S} v\left(T, (N\setminus(T\cup S))\cup L\right) \sum_{K\subset S\setminus L\,,\,K\neq\emptyset} (-1)^{s-k-l} \right] \right\} \end{aligned}$$

We have

$$\sum_{K\subset S\setminus L\,,\,K\neq\emptyset} (-1)^{s-k-l} = \sum_{k=0}^{s-l} \binom{s-l}{k} (-1)^{s-k-l} - (-1)^{s-l} = -(-1)^{s-l} \ .$$

Hence

$$I^v(S) \;=\; \sum_{T\subset N\setminus S} \frac{t!\,(n-s-t)!}{(n-s+1)!} \left\{ \sum_{L\subset S\,,\,L\neq\emptyset} (-1)^{s-l} v\left(L\cup T, N\setminus(T\cup S)\right) \right.$$

$$\left. - \sum_{L\subset S\,,\,L\neq\emptyset} (-1)^{s-l} v\left(T, (N\setminus(T\cup S))\cup L\right) \right\}$$

■

Unlike the importance index, the interaction indices do not look like interaction indices for regular games. $I^v(S)$ is indeed the difference of two terms. This suggests that $I^v(S)$ depicts some complicated behavior involving defenders and defeaters, and can be written in term of more elementary interaction indices. These elementary interactions will be easier to be interpreted. Those are the *bi-interaction indices* defined in the following section.

4 The bi-interaction indices

We aim to define $I^v(S,T) \in \mathbb{R}$. $I^v(S,T)$ is some interaction of players in $S\cup T$ with respect to v, in which there is something special between S and T.

$I^v(S,T)$ satisfies to linearity with respect to the game as for the importance index (see axiom **(l)**) or for the interaction indices (see [9]).

Linearity (L): $v \mapsto I^v(S,T)$ is linear in $\mathcal{G}^{[2]}(N)$.

As for Lemma 1, we obtain :

Lemma 11 $\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ *satisfies* **(L)** *if and only if there exists* $a^{S,T}_{S',T'}$ *for all* $(S',T')\in\mathcal{Q}(N)$ *such that*

$$I^v(S,T) = \sum_{(S',T')\in\mathcal{Q}(N)} a^{S,T}_{S',T'}\, v(S',T')\ .$$

The null player axiom is also necessary. It is satisfied for both the importance index $I^v(i)$ (see axiom **(n)**) and the interaction index $I^v(S)$ for any $i\in S$ (see [9]). If v is null for player i then $I^v(S,T)=0$ whenever i belongs to S or T.

Null player (N): If a player i is null for $v \in \mathcal{G}^{[2]}(N)$ then for any $(S,T) \in \mathcal{Q}(N)$ such that $i \in S \cup T$, we have $I^v(S,T) = 0$.

Let $ST := S \cup T$ and $N_{S,T} := N \setminus (S \cup T)$.

Lemma 12 *$\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ satisfies* **(L)** *and* **(N)** *if and only if there exists $a^{S,T}_{S'\cup L, T'\cup(ST\setminus L)}$ for $(S',T') \in \mathcal{Q}(N_{S,T})$ and $L \subset ST$ such that*

$$I^v(S,T) = \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} \sum_{L\subset ST} a^{S,T}_{S'\cup L, T'\cup(ST\setminus L)} \, \Delta_{L,ST\setminus L} v(S',T') \ .$$

Proof : The *if* part of the lemma is obvious since $\Delta_{L,ST\setminus L}v(S',T')$ clearly vanishes from (2) and the following relation

$$\begin{array}{l} \Delta_{\{i\},\emptyset} v(S',T') = 0 \ \text{ if } i \in S \\ \Delta_{\emptyset,\{i\}} v(S',T') = 0 \ \text{ if } i \in T \end{array}$$

There remains to show the *only if* part. Assume thus that $\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ satisfies **(L)** and **(N)**. By Lemma 11, there exists $a^{S,T}_{S',T'}$ for all $(S',T') \in \mathcal{Q}(N)$ such that

$$I^v(S,T) = \sum_{(S',T')\in\mathcal{Q}(N)} a^{S,T}_{S',T'} \, v(S',T') \ .$$

Assume that $i \in ST$ is null for v. Then by **(N)**

$$\begin{aligned} 0 \ = \ I^v(S,T) &= \sum_{(S',T')\in\mathcal{Q}(N\setminus i)} \Big[a^{S,T}_{S',T'} v(S',T') + a^{S,T}_{S'\cup\{i\},T'} v(S'\cup\{i\},T') \\ &\qquad + a^{S,T}_{S',T'\cup\{i\}} v(S',T'\cup\{i\}) \Big] \\ &= \sum_{(S',T')\in\mathcal{Q}(N\setminus i)} v(S',T') \left[a^{S,T}_{S',T'} + a^{S,T}_{S'\cup\{i\},T'} + a^{S,T}_{S',T'\cup\{i\}} \right] \end{aligned}$$

This holds for any bi-cooperative game v such that i is null for v. Hence, it holds for any restricted game v on $N \setminus \{i\}$. Hence

$$\forall (S',T') \in \mathcal{Q}(N\setminus i) \ , \quad a^{S,T}_{S',T'} = -a^{S,T}_{S'\cup\{i\},T'} - a^{S,T}_{S',T'\cup\{i\}} \qquad (3)$$

We write

$$I^v(S,T) = \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} \sum_{(K,M)\in\mathcal{Q}(ST)} a^{S,T}_{S'\cup K, T'\cup M} \, v(S'\cup K, T'\cup M) \ . \qquad (4)$$

Thanks to (3), we write $a^{S,T}_{S'\cup K,T'\cup M}$ as a sum of terms of the kind $a^{S,T}_{S'\cup L,T'\cup(ST\setminus L)}$ with $L \subset ST$. Set $R = ST \setminus (K \cup M)$. Let us show by induction on r=—R— that

$$a^{S,T}_{S'\cup K,T'\cup M} = (-1)^{s+t-k-m} \sum_{L\subset R} a^{S,T}_{S'\cup K\cup L,T'\cup M\cup(R\setminus L)} \tag{5}$$

For $r = 1$, we set $ST \setminus (K \cup M) = R = \{i\}$. Relation (3) gives

$$a^{S,T}_{S'\cup K,T'\cup M} = -a^{S,T}_{S'\cup K\cup\{i\},T'\cup M} - a^{S,T}_{S'\cup K,T'\cup M\cup\{i\}}$$

which is exactly (5). Assume that (5) holds for $r-1$. Consider then $(K, M) \in \mathcal{Q}(ST)$ with $R = ST \setminus (K \cup M)$ and $|R| = r$. By (3),

$$a^{S,T}_{S'\cup K,T'\cup M} = -a^{S,T}_{S'\cup K\cup\{i\},T'\cup M} - a^{S,T}_{S'\cup K,T'\cup M\cup\{i\}} \ .$$

The two terms in the right hand side can be computed thanks to the induction assumption :

$$\begin{aligned} a^{S,T}_{S'\cup K,T'\cup M} &= -(-1)^{s+t-k-m-1} \sum_{L\subset R\setminus\{i\}} a^{S,T}_{S'\cup K\cup L\cup\{i\},T'\cup M\cup(R\setminus(L\cup\{i\}))} \\ &\quad -(-1)^{s+t-k-m-1} \sum_{L\subset R\setminus\{i\}} a^{S,T}_{S'\cup K\cup L,T'\cup M\cup\{i\}\cup(R\setminus(L\cup\{i\}))} \\ &= (-1)^{s+t-k-m} \sum_{L\subset R} a^{S,T}_{S'\cup K\cup L,T'\cup M\cup(R\setminus L)} \end{aligned}$$

Hence (5) holds for any $r \in 1, \dots, s + t$.

Plugging (5) into (4), we get

$$
\begin{aligned}
I^v(S,T) &= \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} \sum_{(K,M)\in\mathcal{Q}(ST)} v(S'\cup K, T'\cup M) \\
&\quad \times \left[\sum_{L\subset R=ST\setminus(K\cup M)} (-1)^{s+t-k-m} a^{S,T}_{S'\cup K\cup L, T'\cup M\cup(R\setminus L)} \right] \\
&= \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} \sum_{(K,M)\in\mathcal{Q}(ST)} v(S'\cup K, T'\cup M) \\
&\quad \times \left[\sum_{L\subset R=ST\setminus(K\cup M)} (-1)^{s+t-k-m} a^{S,T}_{S'\cup(K\cup L), T'\cup(ST\setminus(K\cup L))} \right] \\
&= \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} \sum_{(K,M)\in\mathcal{Q}(ST)} v(S'\cup K, T'\cup M) \\
&\quad \times \left[\sum_{L'\subset ST \,\mid\, K\subset L', M\subset ST\setminus L'} (-1)^{s+t-k-m} a^{S,T}_{S'\cup L', T'\cup(ST\setminus L')} \right] \\
&= \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} \sum_{L'\subset ST} a^{S,T}_{S'\cup L', T'\cup(ST\setminus L')} \\
&\quad \times \left[\sum_{K\subset L', M\subset ST\setminus L'} (-1)^{(l'-k)+(s+t-l'-m)} v(S'\cup K, T'\cup M) \right] \\
&= \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} \sum_{L'\subset ST} a^{S,T}_{S'\cup L', T'\cup(ST\setminus L')} \Delta_{L', ST\setminus L'} v(S',T')
\end{aligned}
$$

■

Now, we shall give some specificity about $I^v(S,T)$, in particular, regarding the meaning of the two indices. We want to express that $I^v(S,T)$ is the interaction index of v with respect to $S\cup T$ in the defender part for players in S and in the defeater part for players in T. So, no player of S shall become defeater, and no player of T shall become defender. $I^v(S,T)$ shall contain only terms of the bi-cooperative game belonging to

$$\mathcal{Q}_{S,T}(N) = \{(S',T')\in\mathcal{Q}(N) \text{ such that } S'\cap T=\emptyset \text{ and } T'\cap S=\emptyset\} \ .$$

Hence, we have the following axiom :

> **Sign (Sg)**: The terms of $v\in\mathcal{G}^{[2]}(N)$ that appear in $I^v(S,T)$ belong to $\mathcal{Q}_{S,T}(N)$.

Lemma 13 *$\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ satisfies* **(L)**, **(N)** *and* **(Sg)** *if and only if there exists $a^{S,T}_{S',T'}$ for $(S',T')\in\mathcal{Q}(N_{S,T})$ such that*

$$I^v(S,T)=\sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{S,T}_{S\cup S',T\cup T'}\Delta_{S,T}v(S',T') \ .$$

Proof : The *if* part of the proof is obvious and left to the reader.

Assume that $\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ satisfies **(L)**, **(N)** and **(Sg)**. By Lemma 12,

$$\begin{aligned} I^v(S,T) \;=\; & \sum_{(S',T')\in\mathcal{Q}(N_{S,T})}\sum_{L\subset ST} a^{S,T}_{S'\cup L,T'\cup(ST\setminus L)} \\ & \times\left[\sum_{S''\subset L,T''\subset ST\setminus L}(-1)^{(l-s'')+(s+t-l-t'')}v(S'\cup S'',T'\cup T'')\right]\end{aligned}$$

By **(Sg)**, one must have that $(S'\cup S'',T'\cup T'')\in\mathcal{Q}_{S,T}(N)$, that is to say that $S''\cap T=\emptyset$ and $T''\cap S=\emptyset$. This must hold for any $S''\subset L$ and $T''\subset ST\setminus L$. Consider first the case when $L\neq S$. We have the alternative.

- $L\cap T\neq\emptyset$. Taking, in the sum, the case when $S''=L$, the first condition is not fulfilled.
- $L\subset S$ $(L\neq S)$.Then $(ST\setminus L)\cap S\neq\emptyset$. Taking $T''=ST\setminus L$, the second condition is not satisfied.

Hence, if $L\neq S$, there exists at least one couple (S'',T'') in the sum such that $(S'\cup S'',T'\cup T'')\notin\mathcal{Q}_{S,T}(N)$. Therefore, $a^{S,T}_{S'\cup L,T'\cup(ST\setminus L)}=0$ whenever $L\neq S$. If $L=S$, it is easy to see that $(S'\cup S'',T'\cup T'')\notin\mathcal{Q}_{S,T}(N)$ for any $S''\subset L$ and $T''\cap S=\emptyset$. ■

Let us consider now the mirror operator

$$\Pi_A(S,T):=((S\cap A)\cup(T\setminus A),(S\setminus A)\cup(T\cap A)) \ .$$

Define $\Pi_A\circ v$ by

$$\Pi_A\circ v(S,T)=v\left(\Pi_A(S,T)\right) \ .$$

Symmetry implied by Π_A on variables in $N\setminus A$ yields non-increasingness of $\Pi_A\circ v$ w.r.t. variables in $N\setminus A$. The defenders and the defeaters within $N\setminus A$ are switched. Consider first the case when $N\setminus A$ is a singleton, say $A = N \setminus \{i\}$. Since the defender and the defeater part regarding i are switched, the interaction behavior changes sign. More precisely, it is natural that

$$I^{\Pi_{N\setminus\{i\}}\circ v}(\Pi_{N\setminus\{i\}}(S,T)) = -I^v(S,T) \quad \text{for any } (S,T) \in \mathcal{Q}(N) \ .$$

When more variables are concerned by the symmetry, one gets

$$I^{\Pi_A \circ v}\left(\Pi_A(S,T)\right) = (-1)^{|N\setminus A|} I^v(S,T) \ .$$

In particular, for $A = N\setminus T$, we obtain the following axiom since $\Pi_{N\setminus T}(S,T) = (S \cup T, \emptyset)$

> **Mirror (M)**: For any $v \in \mathcal{G}^{[2]}(N)$ and $(S,T) \in \mathcal{Q}(N)$, we have $I^{\Pi_{N\setminus T}\circ v}(S \cup T, \emptyset) = (-1)^t I^v(S,T)$.

This axiom is related to **(m)**.

Lemma 14 *For* $(K,L) \in \mathcal{Q}(N_{S,T})$

$$\Delta_{S\cup T,\emptyset} \Pi_{N\setminus T} \circ v(K,L) = \Delta_{S,T} v(K,L) \ .$$

Proof : We have

$$\begin{aligned}
\Delta_{S\cup T,\emptyset} \Pi_{N\setminus T} \circ v(K,L) &= \sum_{W\subset ST} (-1)^{s+t-w} \Pi_{N\setminus T} \circ v(K \cup W, L) \\
&= \sum_{W\subset ST} (-1)^{s+t-w} v\left(\Pi_{N\setminus T}(K \cup W, L)\right) \\
&= \sum_{W\subset ST} (-1)^{s+t-w} v(K \cup (W \cap S), L \cup (W \cap T)) = \Delta_{S,T} v(K,L) \ .
\end{aligned}$$

■

Lemma 15 $\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ *satisfies* **(L)**, **(N)**, **(Sg)** *and* **(M)** *if and only if there exists* $a^{S,T}_{S',T'}$ *for* $(S',T') \in \mathcal{Q}(N_{S,T})$ *such that*

$$I^v(S,T) = \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{S,T}_{S\cup S',T\cup T'} \Delta_{S,T} v(S',T') \ ,$$

with

$$\forall (S',T') \in \mathcal{Q}(N_{S,T}) \quad , \quad a^{S,T}_{S\cup S',T\cup T'} = (-1)^t a^{S\cup T,\emptyset}_{S\cup T\cup S',T'} \ .$$

Proof : The *if* part of the proof is obvious and left to the reader.

Assume that $\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ satisfies **(L)**, **(N)**, **(Sg)** and **(M)**. By Lemma 13, there exists $a^{S,T}_{S',T'}$ for $(S',T')\in\mathcal{Q}(N_{S,T})$ such that

$$I^v(S,T) = \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{S,T}_{S\cup S',T\cup T'} \Delta_{S,T} v(S',T') \ .$$

We have

$$I^{\Pi_{N\setminus T}\circ v}(S\cup T,\emptyset) = \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{S\cup T,\emptyset}_{ST\cup S',T'} \Delta_{ST,\emptyset} \Pi_{N\setminus T}\circ v(S',T') \ .$$

By Lemma 14,

$$I^{\Pi_{N\setminus T}\circ v}(S\cup T,\emptyset) = \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{S\cup T,\emptyset}_{ST\cup S',T'} \Delta_{S,T} v(K,L) \ .$$

By **(M)**,

$$I^{\Pi_{N\setminus T}\circ v}(S\cup T,\emptyset) = (-1)^t I^v(S,T) = (-1)^t \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{S,T}_{S\cup S',T\cup T'} \Delta_{S,T} v(S',T') \ .$$

Hence for any $(S',T')\in\mathcal{Q}(N_{S,T})$, $a^{S,T}_{S\cup S',T\cup T'} = (-1)^t a^{S\cup T,\emptyset}_{S\cup T\cup S',T'}$. ∎

Interesting particular cases of bi-cooperative games are the ones taking the following form $v(S,T) = \nu_1(S) - \nu_2(T)$ where ν_1 and ν_2 are regular games. This representation is known in decision under risk or uncertainty as *Cumulative Prospect Theory* (CPT) [10]. This representation contains the two classical ways to see regular games as degenerate bi-cooperative games. Consider the interpretation in the framework of ternary voting games. In the first way, the abstentionists are conflated to the *'no'* voters. This can be modeled in the following way : $v(S,T) = \nu_1(S)$. In the second way, the abstentionists are conflated to the *'yes'* voters. This yields $v(S,T) = \overline{\nu_2}(N\setminus T) = \nu_2(N) - \nu_2(T)$. A straightforward calculation shows that the interaction index for a bi-cooperative game of the CPT form is equal to

$$I^v(S) = I^{\nu_1}(S) - (-1)^s I^{\overline{\nu_2}}(S) \ ,$$

where I^{ν_1} and $I^{\overline{\nu_2}}$ are the regular interaction indices [9]. On the other hand, by Lemma 13, $I^v(S,\emptyset)$ contains only terms of the form $\Delta_{S,\emptyset} v$. Clearly, the ν_2 part of v disappears in this discrete derivative. So, $I^v(S,\emptyset)$ is an interaction depending only on ν_1. Since ν_1 is a regular game, this implies that

$I^v(S,\emptyset)$ should be equal to $I^{\nu_1}(S)$. With similar arguments, one comes to the conclusion that $I^v(\emptyset,S)$ should be equal to $I^{\overline{\nu_2}}(S)$. This gives

$$I^v(S) = I^v(S,\emptyset) - (-1)^s I^v(\emptyset,S) \ ,$$

whenever v takes the CPT form.

$I^v(S,\emptyset)$ is the interaction index localized in $\mathcal{Q}_{S,\emptyset}(N)$, whereas $I^v(\emptyset,S)$ is the interaction index localized in $\mathcal{Q}_{\emptyset,S}(N)$. It is reasonable to say that the overall interaction $I^v(S)$ is the difference between those two interactions, as stated in the following axiom. This decomposition looks a little like a discrete derivative.

Decomposition (D): For any $v \in \mathcal{G}^{[2]}(N)$ and $S \subset N$, we have $I^v(S) = I^v(S,\emptyset) - (-1)^s I^v(\emptyset,S)$.

Theorem 1 $\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ *satisfies* **(L)**, **(N)**, **(Sg)**, **(M)** *and* **(D)** *if and only if*

$$I^v(S,T) = (-1)^t \sum_{K\subset N\setminus(S\cup T)} \frac{(n-s-t-k)!k!}{(n-s-t+1)!} \Delta_{S,T} v(K, N\setminus(S\cup T\cup K)) \ .$$

Proof : The *if* part of the proof is obvious and left to the reader.

Assume that $\{I^v(S,T)\}_{(S,T)\in\mathcal{Q}(N)}$ satisfies **(L)**, **(N)**, **(Sg)**, **(M)** and **(D)**. By Lemma 13, there exists $a^{S,T}_{S',T'}$ for $(S',T')\in\mathcal{Q}(N_{S,T})$ such that

$$I^v(S,T) = \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{S,T}_{S\cup S',T\cup T'} \Delta_{S,T} v(S',T') \ .$$

By **(D)**,

$$\begin{aligned}
&I^v(S,\emptyset) - (-1)^s I^v(\emptyset,S)\\
&\quad = \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{S,\emptyset}_{S\cup S',T'} \Delta_{S,\emptyset} v(S',T') - (-1)^s \sum_{(S',T')\in\mathcal{Q}(N_{S,T})} a^{\emptyset,S}_{S',S\cup T'} \Delta_{\emptyset,S} v(S',T')\\
&\quad = I^v(S) = \sum_{K\subset N\setminus S} \frac{(n-s-k)!k!}{(n-s+1)!}\\
&\qquad\qquad \times [\Delta_{S,\emptyset} v(K, N\setminus(S\cup K)) - \Delta_{\emptyset,S} v(K, N\setminus(S\cup K))] \qquad (6)
\end{aligned}$$

On the other hand,

$$\begin{aligned}\Delta_{S,\emptyset}v(S',T') &= \sum_{K\subset S}(-1)^{s-k}v(K\cup S',T')\\ \Delta_{\emptyset,S}v(S',T') &= \sum_{K\subset S}(-1)^{s-k}v(S',K\cup T')\end{aligned}$$

So, $\Delta_{S,\emptyset}v(S',T')$ contains the term $v(S\cup S',T')$. It is not contained in any other term $\Delta_{S,\emptyset}v(S'',T'')$ (with $(S'',T'')\in\mathcal{Q}(N\setminus S)\setminus\{(S',T')\}$), and in any term $\Delta_{\emptyset,S}v(S'',T'')$ (with $(S'',T'')\in\mathcal{Q}(N\setminus S)$). Similarly, $\Delta_{\emptyset,S}v(S',T')$ contains the term $v(S',S\cup T')$. It is not contained in any other term $\Delta_{\emptyset,S}v(S'',T'')$ (with $(S'',T'')\in\mathcal{Q}(N\setminus S)\setminus\{(S',T')\}$), and in any term $\Delta_{S,\emptyset}v(S'',T'')$ (with $(S'',T'')\in\mathcal{Q}(N\setminus S)$). From these arguments, we can infer from (6) that

$$\begin{aligned}a^{S,\emptyset}_{S\cup S',T'} &= \begin{cases}\frac{(n-s-s')!s'!}{(n-s+1)!} & \text{if } T'=N\setminus(S\cup S')\\ 0 & \text{otherwise}\end{cases}\\ a^{\emptyset,S}_{S',S\cup T'} &= \begin{cases}(-1)^s\frac{(n-s-s')!s'!}{(n-s+1)!} & \text{if } T'=N\setminus(S\cup S')\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Hence

$$I^v(S,\emptyset)=\sum_{K\subset N\setminus S}\frac{(n-s-k)!k!}{(n-s+1)!}\Delta_{S,\emptyset}v(K,N\setminus(S\cup K))\ .$$

Finally, from **(M)**

$$I^v(S,T)=(-1)^tI^{\Pi_{N\setminus T}\circ v}(S\cup T,\emptyset)=(-1)^t\sum_{K\subset N_{S,T}}\frac{(n-s-t-k)!k!}{(n-s-t+1)!}\Delta_{S\cup T,\emptyset}\Pi_{N\setminus T}\circ v(K,L)\ .$$

By Lemma 14

$$I^v(S,T)=(-1)^t\sum_{K\subset N_{S,T}}\frac{(n-s-t-k)!k!}{(n-s-t+1)!}\,\Delta_{S,T}v(K,N\setminus(S\cup T\cup K))\ .$$

■

Let us now give the interpretation of bi-interactions. $I^v(S,T)$ is the interaction index of v with respect to $S\cup T$ in the defender part for players of S and the defeater part for players of T.

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