Bi-Capacities

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Abstract

We introduce bi-capacities as a natural generalization of capacities (or fuzzy measures) through the identity of Choquet integral of binary alternatives with fuzzy measures. We examine the underlying structure and derive the Möbius transform of bi-capacities. Next, the Choquet and Sugeno integrals w.r.t bi-capacities are introduced. It is shown that symmetric and asymmetric integrals are recovered. Lastly, we introduce the Shapley value and interaction indices. It is seen that besides a generalization based on the classical definitions, a definition involving two arguments is natural.

***Keywords*— bi-capacity, Choquet integral, Möbius transform, Shapley value, interaction index.**

1 Introduction

Capacities [1] (or fuzzy measures [15], non-additive measures [3]), and integrals w.r.t capacities such as the Choquet integral [1] and the Sugeno integral [15], have become a central tool in decision making, extending e.g. expected utility models and linear models of multiattribute utility theory. While the so-called Choquet expected utility models are among the most general models for preference representation when utility functions are positive, the introduction of negative quantities for utility functions makes possible several extension of the usual Choquet expected utility model. Up to now, the most general extension is called the CPT model (Cumulative Prospect Theory [16]), and is a difference of two Choquet integrals.

More specifically, let us denote by ν_1, ν_2 two capacities on a finite universe N of n elements. For any real-valued function on N, the CPT model is expressed as:

$$\mathrm{CPT}_{\nu_1,\nu_2}(f) := \mathcal{C}_{\nu_1}(f^+) - \mathcal{C}_{\nu_2}(f^-)$$

where $\mathcal{C}_{\nu_i}$ is the Choquet integral with respect to the capacity ν_i, and $f^+ := f \vee 0$, $f^- = (-f)^+$. We denote by 1_A the characteristic function of A, for any $A \subset N$. We call these functions *binary functions*, as they take only values 0 and 1. It is known that for any $A \subset N$, any capacity ν, we have $\mathcal{C}_\nu(1_A) = \nu(A)$. Hence the capacity is uniquely determined by giving the integral (or expected utility) of all binary functions. Let us consider ternary functions, i.e. functions valued on $\{-1, 0, 1\}$, which we express under the form $(1_A, -1_B)$, for $A, B \subset N$, $A \cap B = \emptyset$. Observe that

$$\mathrm{CPT}_{\nu_1,\nu_2}(1_A, -1_B) = \nu_1(A) - \nu_2(B),$$

which entails that, if ν_1, ν_2 are given, we have no freedom for determining the utility of a ternary alternative (function), since this value is determined from the utility of two binary alternatives.

In order to get rid of this limitation, we introduce bi-capacities as the value assigned to a ternary function.

The report presents first results on bi-capacities, their structure and machinery. We assume basic knowledge on capacities and Choquet integral (see e.g. [9] for background). To avoid heavy notations, we will often omit braces and commas to denote sets. Also, the cardinal of a set is denoted by the corresponding small letter, e.g. $|N| = n$.

2 Bi-capacities

The definition is given through the following: an act with scores equal to 1 on $A \subset N$, to -1 on $B \subset N$ has an overall utility denoted $v(A, B)$. By convention, $v(\emptyset, \emptyset) = 0$, $v(N, \emptyset) = 1$ and $v(\emptyset, N) = -1$. Due to basic assumptions in decision making (dominance), we have

$$v(A, B) \leq v(C, B) \text{ if } A \subset C, \quad v(A, B) \geq v(A, D) \text{ if } B \subset D.$$

This leads to the following definition. We denote $\mathcal{Q}(N) := \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) | A \cap B = \emptyset\}$.

Definition 1 *A function $v : \mathcal{Q}(N) \longrightarrow \mathbb{R}$ is a* bi-capacity *if it satisfies:*

(i) $v(\emptyset, \emptyset) = 0$

(ii) $A \subset B$ *implies* $v(A, \cdot) \leq v(B, \cdot)$ *and* $v(\cdot, A) \geq v(\cdot, B)$.

In addition, v is normalized *if* $v(N, \emptyset) = 1 = -v(\emptyset, N)$.

In the sequel, unless otherwise specified, we will consider that bi-capacities are normalized. Note that the definition implies that $v(\cdot, \emptyset) \geq 0$ and $v(\emptyset, \cdot) \leq 0$.

An interesting particular case is when left and right part can be separated. We say that a bi-capacity is of the *CPT type* (refering to Cumulative Prospect Theory [16]) if there exists two (normalized) capacities ν_1, ν_2 such that

$$v(A, B) = \nu_1(A) - \nu_2(B), \forall (A, B) \in \mathcal{Q}(N).$$

When $\nu_1 = \nu_2$, we say that the bi-capacity is *symmetric*, and *asymmetric* when $\nu_2 = \overline{\nu}_1$.

By analogy with the classical case, a bi-capacity is said to be *additive* if it satisfies for all $(A, B) \in \mathcal{Q}(N)$:

$$v(A, B) = \sum_{i \in A} v(i, \emptyset) + \sum_{i \in B} v(\emptyset, i). \tag{1}$$

They are defined by a "positive" distribution and a "negative" distribution, and are clearly of the CPT type. More generally, decomposable bi-capacities can be defined as well, using t-conorms. However, the problem arises of combining positive and negative quantities. For a given t-conorm S, it is known that extensions of S on $[-1, 1]$ so that the structure is a group can be built only if S is a strict t-conorm [6] [......detail if necessary......].

Definition 2 *Let S be a strict t-conorm. A bi-capacity is said to be S-*decomposable *if it satisfies:*

$$v(A, B) = (S_{i \in A} v(i, \emptyset)) \ominus_S (S_{i \in B}(-v(\emptyset, i)))$$

where $\ominus_S$ is the S-difference (see [6]).

However the case of maximum remains interesting since despite its extension fails to build a group only due to a lack of associativity, associativity holds when negative and positive quantities are merged separately [4, 5].

Definition 3 *A bi-capacity is said to be* max-decomposable *or to be a* bi-measure of possibility *if*

$$v(A, B) = \big[\underset{i \in A}{\varovee} v(i, \emptyset) \big] \varovee \big[\underset{i \in B}{\varovee} v(\emptyset, i) \big],$$

where $\varovee$ is the symmetric maximum [4] defined by

$$a \varovee b := \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and either } |a| \vee |b| = -a \\ & \text{or } \ = -b \\ 0 & \text{if } b = -a \\ |a| \vee |b| & \text{else.} \end{cases}$$

To avoid heavy notations, we will often omit braces and commas to denote sets. For example, $\{i\}, \{i, j\}, \{1, 2, 3\}$ are denoted respectively $i, ij, 123$.

3 The structure of $\mathcal{Q}(N)$

We study in this section the structure of $\mathcal{Q}(N)$. From its definition, $\mathcal{Q}(N)$ is ismorphic to the set of mappings from N to $\{-1, 0, 1\}$, hence $|\mathcal{Q}(N)| = 3^n$. Also, any element (A, B) in $\mathcal{Q}(N)$ can be denoted $(x_1, \dots, x_n)$, with $x_i \in \{-1, 0, 1\}$, with $x_i = 1$ if $i \in A$, $x_i = -1$ if $i \in B$, and 0 otherwise.

As a preliminary remark, $\mathcal{Q}(N)$ is a subset of $\mathcal{P}(N)^2$, and so can be represented in a matrix form, using some total order on $\mathcal{P}(N)$. A natural order is the binary order, obtained by ordering in an increasing sequence the integers coding the elements of $\mathcal{P}(N)$:

$\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}$, etc. Using this order, the matrix has a fractal structure with generating pattern

$$\begin{array}{cc} \times & \times \\ \times & \end{array}$$

We give below the matrix obtained with $n = 3$.

$$\begin{array}{c} \\ \emptyset \\ 1 \\ 2 \\ 12 \\ 3 \\ 13 \\ 23 \\ 123 \end{array} \begin{array}{c} \begin{array}{cccccccc} \emptyset & 1 & 2 & 12 & 3 & 13 & 23 & 123 \end{array} \\ \left[\begin{array}{cccccccc} \times & \times & \times & \times & \times & \times & \times & \times \\ \times & & \times & & \times & & \times & \\ \times & \times & & & \times & \times & & \\ \times & & & & \times & & & \\ \times & \times & \times & \times & & & & \\ \times & & \times & & & & & \\ \times & \times & & & & & & \\ \times & & & & & & & \end{array}\right] \end{array}$$

As for $\mathcal{P}(N)$, it is convenient to define a total order on $\mathcal{Q}(N)$, so as to reveal structures. A natural one is to use a ternary coding. Several are possible, but it seems that the most suitable one is to code (considering elements of $\mathcal{Q}(N)$ as mappings valued on $\{-1, 0, 1\}$) -1 by 0, 0 by 1, and 1 by 2. The increasing sequence of integers in ternary code is 0, 1, 2, 10, 11, 12, 20 etc., which leads to the following order of elements of $\mathcal{Q}(N)$:

$$\cdots (2,3)\ (12,3)\ \boxed{(\emptyset,12)\ (\emptyset,2)\ (1,2)\ \boxed{(\emptyset,1)\ \boxed{(\emptyset,\emptyset)}\ (1,\emptyset)}\ (2,1)\ (2,\emptyset)\ (12,\emptyset)}\ (3,12)\ (3,2) \cdots$$

Again, we remark a fractal structure, which is enhanced by boxes: the $(k+1)$th box is built form the kth box by adding to its elements (of $\mathcal{Q}(N)$) element k of N, either to their left part, or to their right part.

It is easy to see that $\mathcal{Q}(N)$ is a lattice, when equipped with the following order: $(A, B) \leq (C, D)$ if $A \subset C$ and $B \supset D$. Supremum and infimum are respectively

$$\begin{aligned} (A, B) \vee (C, D) &= (A \cup C, B \cap D) \\ (A, B) \wedge (C, D) &= (A \cap C, B \cup D). \end{aligned}$$

These are elements of $\mathcal{Q}(N)$ since $(A \cup C) \cap (B \cap D) = \emptyset$ and $(A \cap C) \cap (B \cup D) = \emptyset$. Top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$. Notice that a bi-capacity is an order-preserving mapping from $\mathcal{Q}(N)$ to $\mathbb{R}$. We call *vertices* of $\mathcal{Q}(N)$ any element (A, B) such that $A \cup B = N$, since they coincide with the vertices of $[0,1]^n$. We give in figures 1 and 2 the Hasse diagram of $(\mathcal{Q}(N), \leq)$ for $n = 2$ and $n = 3$.

$\mathcal{Q}(N)$ is in fact the lattice 3^n. It is formed by 2^n Boolean sub-lattices 2^n: each sub-lattice corresponds to a given partition of N into two parts, one for positive scores, the other for negative ones, which contain all subsets of non-zero scores, including the empty set. Hence, all these sub-lattices have as a common point $(\emptyset, \emptyset)$.

For any ordered pair $((A, B), (A \cup D, B \setminus C))$ of $\mathcal{Q}(N)$ with $C \subset B$ and $D \subset (N \setminus (A \cup B)) \cup C$ the interval $[(A, B), (A \cup D, B \setminus C)]$ is a sub-lattice of type $2^k \times 3^l$, with $k = |C \Delta D|$,

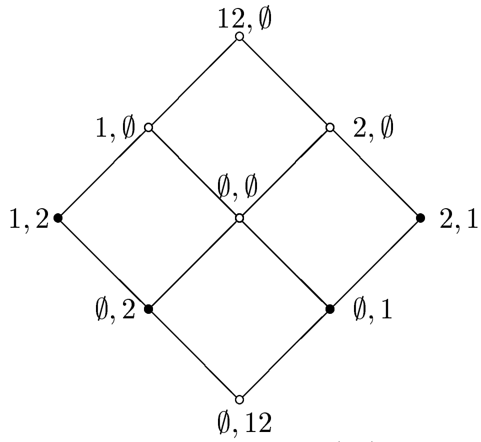

Figure 1: The lattice $\mathcal{Q}(N)$ for $n = 2$

and $l = |C \cap D|$. As a particular case, a sub-lattice of type 2^k is obtained if $C \cap D = \emptyset$, and of type 3^l if $C = D$.

We use the notation

$$\uparrow (A, B) := \{(C, D) \in \mathcal{Q}(N) \,|\, (C, D) \geq (A, B)\}$$

to denote the up-set of (A, B).

Let us remark that the nodes of $\mathcal{Q}(N)$ appear in a rather unnatural way. It is possible to have a more natural structure if we replace each node (A, B) by (A, B^c). Let us call $(\mathcal{Q}^*(N), \leq^*)$ this new lattice. A node (A, B) in $\mathcal{Q}^*(N)$ is such that $A \subset B$, and A is the set of scores equal to 1, while B is the set of scores being equal to 0 or 1. We have

$$\begin{aligned}
(A, B) \leq^* (C, D) &\text{ if and only if } A \subset C \text{ and } B \subset D \\
(A, B) \vee^* (C, D) &= (A \cup C, B \cup D) \\
(A, B) \wedge^* (C, D) &= (A \cap C, B \cap D).
\end{aligned}$$

Hence, $\leq^*$ is simply the product order on $\mathcal{P}(N)^2$. Figures 3 and 4 show the Hasse diagram of $(\mathcal{Q}^*(N), \leq)$ for $n = 2$ and $n = 3$.

Let us give some properties of $\mathcal{Q}(N)$ (they are the same for $\mathcal{Q}^*(N)$). Since 3^n is a product of distributive lattices, it is itself distributive (see e.g. [2]). However it is not complemented, since for example $(\emptyset, \emptyset)$ has no complement (b is the complement of a if $a \wedge b = \bot$ and $a \vee b = \top$). Hence, $(\mathcal{Q}(N), \leq)$ is not a Boolean lattice, and so is not isomorphic to the power set of some set.

Nevertheless, it is possible to give a simpler representation of $\mathcal{Q}(N)$. We recall some definitions and a fundamental result of lattice theory [2].

Definition 4 *Let $(L, \leq)$ be a lattice. An element $x \in L$ is* $\vee$-irreducible *if $x \neq \bot$ and $x = a \vee b$ implies $x = a$ or $x = b$, $\forall a, b \in L$.*

In a finite lattice, x is $\vee$-irreducible if it has only one predecessor.

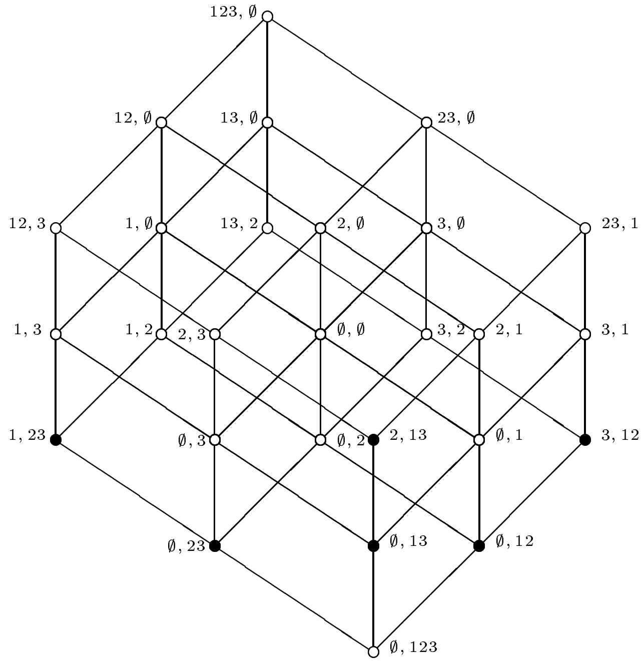

Figure 2: The lattice $\mathcal{Q}(N)$ for $n = 3$

Definition 5 *Let $(P, \leq)$ be a partially ordered set. $Q \subset P$ is a* downset *of P if $x \in Q$ and $y \leq x$ imply $y \in Q$.*

Proposition 1 *Every finite distributive lattice $(L, \leq)$ is isomorphic to $\mathcal{O}(J)$, where J is the set of the $\vee$-irreducible elements of L, and $\mathcal{O}(J)$ is the set of all down-sets of J.*

It is easy to see that the $\vee$-irreducible elements of $\mathcal{Q}(N)$ are $(\emptyset, i^c)$ and (i, i^c), for all $i \in N$. We have for any $(A, B) \in \mathcal{Q}(N)$,

$$(A, B) = \bigvee_{i \in A} (i, i^c) \vee \bigvee_{j \in N \setminus B} (\emptyset, j^c).$$

In $(\mathcal{Q}^*(N), \leq^*)$, the irreducible elements are $(\emptyset, i)$ and (i, i), $\forall i \in N$. On figures 1 to 4, $\vee$-irreducible elements are indicated by black circles.

This permits to define *layers* in $\mathcal{Q}(N)$ as follows: $(\emptyset, N)$ is the bottom layer (layer 0), the set of all $\vee$-irreducible elements form layer 1, and layer k, for $k = 2, \ldots, n$, contains all elements which can be written as the supremum over exactly k $\vee$-irreducible elements. Layer k is denoted $\mathcal{Q}^{[k]}(N)$, and contains all elements (A, B) such that $|B| = n - k$. The top layer (layer n) is reduced to $(N, \emptyset)$.

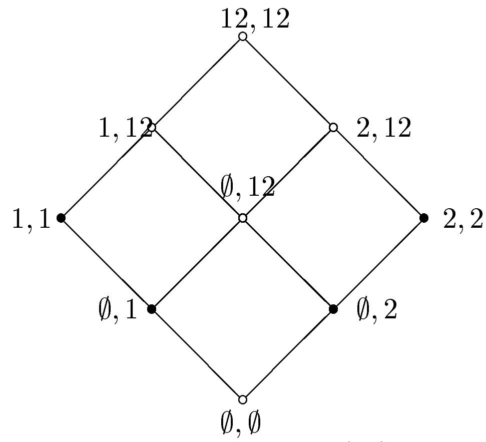

Figure 3: The lattice $\mathcal{Q}^*(N)$ for $n = 2$

4 Möbius transform of bi-capacities

Let us recall some basic facts about the Möbius transform (see [13]). Let us consider f, g two real-valued functions on a locally finite poset $(X, \leq)$ such that

$$g(x) = \sum_{y \leq x} f(y). \tag{2}$$

The solution of this equation in term of g is given through the Möbius function by

$$f(x) = \sum_{y \leq x} \mu(y, x) g(y) \tag{3}$$

where μ is defined inductively by

$$\mu(x, y) = \begin{cases} 1, & \text{if } x = y \\ -\sum_{x \leq t < y} \mu(x, t), & \text{if } x < y \\ 0, & \text{otherwise.} \end{cases}$$

Note that μ depends only on the structure of $(X, \leq)$. When $(X, \leq)$ is a Boolean lattice, e.g. $(\mathcal{P}(N), \subset)$, it is well known that the Möbius function becomes, for any $A, B \in \mathcal{P}(N)$

$$\mu(A, B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subset B \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Observe that this Möbius function has the following property

$$\sum_{A \subset C \subset B} \mu(A, C) = 0, \quad \forall A, B \subset N, A \neq B. \tag{5}$$

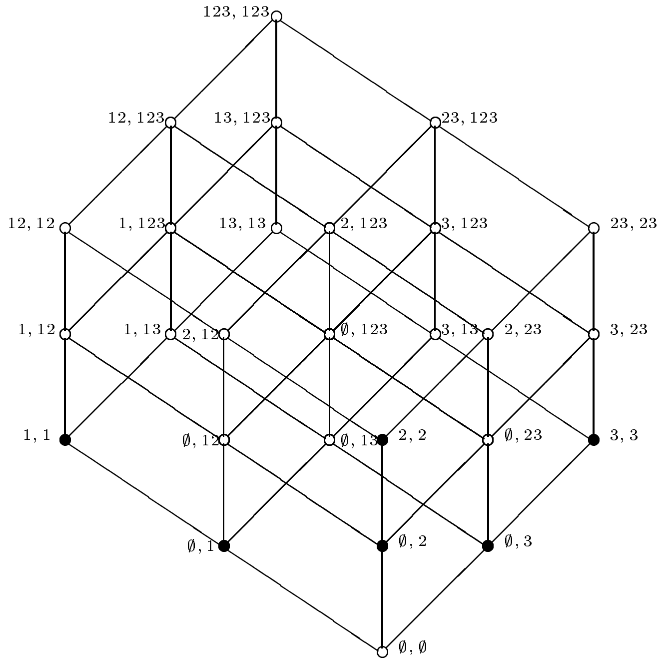

Figure 4: The lattice $\mathcal{Q}^*(N)$ for $n = 3$

Indeed, when $A \subsetneq B$

$$\begin{aligned}\sum_{A\subset C\subset B} \mu(A, C) &= \sum_{A\subset C\subset B} (-1)^{|C\setminus A|} \\ &= \sum_{k=0}^{|B\setminus A|} \binom{|B \setminus A|}{k} (-1)^k \\ &= (1-1)^{|B\setminus A|} = 0.\end{aligned}$$

If g is a capacity, which we denote by v, then f in Eq. (2) is called the *Möbius transform* of v, usually denoted by m or m^v if necessary. Equations (2) and (3) write

$$v(A) = \sum_{B\subset A} m(B) \tag{6}$$

$$m(A) = \sum_{B\subset A} (-1)^{|B\setminus A|} v(B). \tag{7}$$

The Möbius transform is an important concept for capacities, as it can be viewed as the coordinates of v in the basis of unanimity games, defined by

$$u_B(A) = \begin{cases} 1, & \text{if } A \supset B \\ 0, & \text{otherwise.} \end{cases}$$

Then Eq. (6) can be rewritten as

$$v(A) = \sum_{B\subset N} m(B) u_B(A).$$

Our aim is to compute similar formulas for bi-capacities. The first step is to obtain the Möbius function on $\mathcal{Q}(N)$.

Theorem 1 *The Möbius function on $\mathcal{Q}(N)$ is given by, for any $(A,A'),(B,B')\in\mathcal{Q}(N)$*

$$\mu((A,A'),(B,B')) = \begin{cases} (-1)^{|B\setminus A|+|A'\setminus B'|}, & \text{if } (A,A')\leq (B,B') \text{ and } A'\cap B=\emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Proof: We use the fact that if P,Q are posets, then the Möbius function on $P\times Q$ with the product order is the product of the Möbius functions on P and Q [13]. In our case, this gives

$$\mu_{3^n}((x_1,y_1),\dots,(x_n,y_n)) = \prod_{i=1}^n \mu_3(x_i,y_i)$$

where μ_{3^n} is the Möbius function on $\mathcal{Q}(N)=3^n$, μ_3 the Möbius function on $3:=\{-1,0,1\}$, and $(x_1,\dots,x_n),(y_1,\dots,y_n)\in\{-1,0,1\}^n$ correspond to $(A,A'),(B,B')$ respectively. It is easy to see that

$$\mu_3(x_i,y_i) = \begin{cases} 1, & \text{if } x_i=y_i \\ -1, & \text{if } x_i=y_i-1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mu_{3^n}((x_1,y_1),\dots,(x_n,y_n))=0$ iff there is some $i\in N$ such that $\mu_3(x_i,y_i)=0$. This conditions reads $x_i>y_i$ or $x_i=-1,y_i=1$. In term of subsets, this means $(A,A')\not\leq(B,B')$ or $B\cap A'\neq\emptyset$.

We have $\mu_{3^n}((x_1,y_1),\dots,(x_n,y_n))=1$ iff there is no $i\in N$ such that $\mu_3(x_i,y_i)=0$, and the number of $i\in N$ such that $\mu_3(x_i,y_i)=-1$ is even. We examine the second condition. We have:

$$\mu_3(x_i,y_i)=-1 \Leftrightarrow \begin{cases} x_i=0 & \text{and } y_i=1 \\ \text{or} & \\ x_i=-1 & \text{and } y_i=0 \end{cases}$$

which in terms of subsets, reads ($|B\setminus A|=1$ and $|A'\setminus B'|=0$) or ($|B\setminus A|=0$ and $|A'\setminus B'|=1$). Then clearly the second condition is equivalent to $|B\setminus A|+|A'\setminus B'|=2k$. The case $\mu_{3^n}((x_1,y_1),\dots,(x_n,y_n))=-1$ works similarly. ■

Consequently, the Möbius transform of v is expressed by

$$m(A,A') = \sum_{\substack{(B,B')\leq(A,A')\\ B'\cap A=\emptyset}} (-1)^{|A\setminus B|+|B'\setminus A'|}v(B,B') = \sum_{\substack{B\subset A\\ A'\subset B'\subset A^c}} (-1)^{|A\setminus B|+|B'\setminus A'|}v(B,B'). \quad (8)$$

By definition of the Möbius transform, we have

$$v(A,A') = \sum_{(B,B')\leq(A,A')} m(B,B'). \quad (9)$$

As immediate properties of m, let us remark that $m(\emptyset, N) = v(\emptyset, N) = -1$, and $\sum_{(A,B)\in\mathcal{Q}(N)} m(A,B) = v(N,\emptyset) = 1$.

Proceeding as in [8], we may write the Möbius transform into a matrix form, using the total order we have defined on $\mathcal{Q}(N)$. Denoting v, m put in vector form as $v_{(n)}, m_{(n)}$, Eq. (8) can be rewritten as

$$m_{(n)} = T_{(n)} \circ v_{(n)}$$

where $\circ$ is the usual matrix product, and $T_{(n)}$ is the matrix coding the Möbius transform. As with the case of classical capacities, $T_{(n)}$ has an interesting fractal structure, as it can be seen from the case $n = 2$ illustrated below.

$$T_{(2)} = \begin{array}{c} \\ \emptyset,12 \\ \emptyset,2 \\ 1,2 \\ \emptyset,1 \\ \emptyset,\emptyset \\ 1,\emptyset \\ 2,1 \\ 2,\emptyset \\ 12,\emptyset \end{array} \begin{array}{c} \begin{array}{ccccccccc} \emptyset,12 & \emptyset,2 & 1,2 & \emptyset,1 & \emptyset,\emptyset & 1,\emptyset & 2,1 & 2,\emptyset & 12,\emptyset \end{array} \\ \left[\begin{array}{ccccccccc} 1 & & & & & & & & \\ -1 & 1 & & & & & & & \\ & -1 & 1 & & & & & & \\ -1 & & & 1 & & & & & \\ 1 & -1 & & -1 & 1 & & & & \\ & 1 & -1 & & -1 & 1 & & & \\ & & & -1 & & & 1 & & \\ & & & 1 & -1 & & -1 & 1 & \\ & & & & 1 & -1 & & -1 & 1 \end{array}\right] \end{array}$$

The generating element has the form

$$\begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{bmatrix}$$

and is the concatenation of two generating elements $\left[\begin{smallmatrix} 1 & \\ -1 & 1 \end{smallmatrix}\right]$ of the Möbius transform for classical capacities [8].

Let us examine several particular cases of bi-capacities.

Proposition 2 *Let v be a bi-capacity of the CPT type, with $v(A,B) = \nu_1(A) - \nu_2(B)$. Then its Möbius transform is given by:*

$$\begin{aligned} m(A, A^c) &= m^{\nu_1}(A), \quad \forall A \subset N, A \neq \emptyset \\ m(\emptyset, B) &= m^{\overline{\nu_2}}(B^c), \quad \forall B \subsetneq N \\ m(\emptyset, N) &= -1 \\ m(A,B) &= 0, \quad \forall (A,B) \in \mathcal{Q}(N) \textit{ such that } A \neq \emptyset \textit{ and } B \neq A^c. \end{aligned}$$

Proof: Let us consider $A \neq \emptyset$. We have

$$m(A, A') = \sum_{A' \subset B' \subset A^c} (-1)^{|B' \setminus A'|} \left[\sum_{B \subset A} (-1)^{|A \setminus B|} v(B, B') \right].$$

$$\sum_{B\subset A}(-1)^{|A\setminus B|}v(B,B') = \sum_{B\subset A}(-1)^{|A\setminus B|}(\nu_1(B)-\nu_2(B'))$$
$$= \sum_{B\subset A}(-1)^{|A\setminus B|}\nu_1(B) - \nu_2(B')\sum_{B\subset A}(-1)^{|A\setminus B|}$$
$$= \sum_{B\subset A}(-1)^{|A\setminus B|}\nu_1(B) = m^{\nu_1}(A),$$

where we have used (5). Putting in $m(A,A')$ leads to

$$m(A,A') = m^{\nu_1}(A)\sum_{A'\subset B'\subset A^c}(-1)^{|B'\setminus A'|}.$$

Using again (5), the sum is zero unless $A' = A^c$ (only one term in the sum). Hence we get

$$m(A,A') = \begin{cases} m^{\nu_1}(A), & \text{if } A' = A^c \\ 0, & \text{otherwise.} \end{cases}$$

Let us now take $A = \emptyset$. We have:

$$m(\emptyset,A') = \sum_{A'\subset B'\subset N}(-1)^{|B'\setminus A'|}v(\emptyset,B')$$
$$= -\sum_{B'\supset A'}(-1)^{|B'\setminus A'|}\nu_2(B').$$

Let us consider $A' \neq N$, since in this case we know already that $m(\emptyset,N) = -1$. We recall that the co-Möbius transform [....] of a capacity ν is defined by

$$\check{m}^\nu(A) = \sum_{B\supset A^c}(-1)^{|N\setminus B|}\nu(B).$$

We remark that $m(\emptyset,A') = (-1)^{|N\setminus A|+1}\check{m}^{\nu_2}(A^c)$. Using the fact that $\check{m}^\nu(A) = (-1)^{|A|+1}m^{\overline{\nu}}(A)$ for any $A \neq \emptyset$ [...], we get finally $m(\emptyset,A) = m^{\overline{\nu_2}}(A^c)$. ■

We get as immediate corollaries the expression of the Möbius transform of symmetric and asymmetric bi-capacities. Observe in particular that for asymmetric bi-capacities $v(A,B) = \nu(A) - \overline{\nu}(B)$, we have for any $A \neq N$

$$m(\emptyset,A) = m^\nu(A^c).$$

Applying the above result leads easily to the following one.

Proposition 3 *Let v be an additive bi-capacity on $\mathcal{Q}(N)$. Then its Möbius transform is non null only for the $\vee$-irreducible elements and the bottom of $\mathcal{Q}(N)$. Specifically,*

$$m(\emptyset,i^c) = -v(\emptyset,i), \forall i\in N$$
$$m(i,i^c) = v(i,\emptyset), \forall i\in N$$
$$m(\emptyset,N) = -1.$$

The definition of the Möbius transform permits us to introduce *k-additive bi-capacities.*

Definition 6 *A bi-capacity is said to be* k-additive *if its Möbius transform vanishes for all elements (A, B) in $\mathcal{Q}^{[l]}(N)$, for $l = k+1, \dots, n$.*

Equivalently, v is k-additive iff $m(A, B) = 0$ whenever $|B| < n - k$. Clearly, 1-additive bi-capacities coincide with additive bi-capacities.

5 The Choquet and Sugeno integrals with respect to bi-capacities

The expression of the Choquet integral w.r.t a bi-capacity has been introduced axiomatically in [12], see also a presentation based on symmetry considerations in [7]. For any function f on N, we denote by f_i the value $f(i)$, $i \in N$.

Definition 7 *Let v be a bi-capacity and f be a real-valued function on N. The Choquet integral of f w.r.t v is given by*

$$\mathcal{C}_v(f) := \mathcal{C}_{\nu_{N^+}}(|f|)$$

where ν_{N^+} is a real-valued set function on N defined by

$$\nu_{N^+}(C) := v(C \cap N^+, C \cap N^-),$$

and $N^+ := \{i \in N | f_i \geq 0\}$, $N^- = N \setminus N^+$.

Observe that we have $\mathcal{C}_v(1_A, -1_B) = v(A, B)$ for any $(A, B) \in \mathcal{Q}(N)$.

$\mathcal{C}_v(f)$ can be rewritten as:

$$\mathcal{C}_v(f) = \sum_{i=1}^{n} |f_{\sigma(i)}| \Big[v(A_{\sigma(i)} \cap N^+, A_{\sigma(i)} \cap N^-) - v(A_{\sigma(i+1)} \cap N^+, A_{\sigma(i+1)} \cap N^- \Big] \tag{10}$$

where $A_{\sigma(i)} := \{\sigma(i), \dots, \sigma(n)\}$, and σ is a permutation on N so that $|f_{\sigma(1)}| \leq \cdots \leq |f_{\sigma(n)}|$. The above formula is very similar to the one proposed by Greco *et al.* [11].

The following result shows that our definition encompasses the CPT model, and consequently the symmetric and asymmetric Choquet integrals.

Proposition 4 *If v is of the CPT type, with $v(A, B) = \nu_1(A) - \nu_2(B)$, the Choquet integral reduces to*

$$\begin{aligned}
\mathcal{C}_v(f) &= \sum_{i=1}^{n} f^+_{\sigma(i)} \Big[\nu_1(A_{\sigma(i)} \cap N^+) - \nu_1(A_{\sigma(i+1)} \cap N^+)] \\
&\quad - \sum_{i=1}^{n} f^-_{\sigma(i)} \Big[\nu_2(A_{\sigma(i)} \cap N^-) - \nu_2(A_{\sigma(i+1)} \cap N^-)] \\
&= \mathcal{C}_{\nu_1}(f^+) - \mathcal{C}_{\nu_2}(f^-).
\end{aligned}$$

Proof: Using the definition of v, and splitting N in N^+, N^-, (10) becomes:

$$\mathcal{C}_v(f) = \sum_{\sigma(i)\in N^+} f_{\sigma(i)}\Big[\nu_1(A_{\sigma(i)}\cap N^+) - \nu_2(A_{\sigma(i)}\cap N^-) - \nu_1(A_{\sigma(i+1)}\cap N^+) + \nu_2(A_{\sigma(i+1)}\cap N^-)\Big]$$
$$- \sum_{\sigma(i)\in N^-} f^-_{\sigma(i)}\Big[\nu_1(A_{\sigma(i)}\cap N^+) - \nu_2(A_{\sigma(i)}\cap N^-) - \nu_1(A_{\sigma(i+1)}\cap N^+) + \nu_2(A_{\sigma(i+1)}\cap N^-)\Big].$$

If $\sigma(i) \in N^+$, then $A_{\sigma(i)} \cap N^- = A_{\sigma(i+1)} \cap N^-$, and if $\sigma(i) \in N^-$, we have $A_{\sigma(i)} \cap N^+ = A_{\sigma(i+1)} \cap N^+$. Substituting in the above expression leads to the first relation. Let us denote by σ^+, σ^- the permutations on N such that f^+, f^- become non decreasing. Observe that $A_{\sigma^+(i)} = A_{\sigma(i)} \cap N^+$ and $A_{\sigma^-(i)} = A_{\sigma(i)} \cap N^-$, which proves the second relation. ■

A similar construction can be done with the Sugeno integral. Let $\mathcal{S}_\nu(f)$ denotes the Sugeno integral of $f : N \longrightarrow [0,1]$ w.r.t. a capacity ν.

Definition 8 *Let v be a bi-capacity and $f : N \longrightarrow [-1,1]$. The Sugeno integral of f w.r.t v is defined by*

$$\mathcal{S}_v(f) := \mathcal{S}_{\nu_{N^+}}(|f|)$$

with the same notations as above.

However, since ν_{N^+} can assume negative values, the usual definition of Sugeno integral has to be replaced by the following one [5]

$$\mathcal{S}_\nu(f) := \langle \bigotimes_{i=1}^{n} [f_{\sigma(i)} \oslash \nu(A_{\sigma(i)})]\rangle$$

where f is assumed to be non negative, ν is any real-valued set function such that $\nu(\emptyset) = 0$, and σ is a permutation on N such that f becomes non decreasing. Also, $\otimes$ is the symmetric maximum, $\oslash$ the symmetric minimum defined by

$$a \oslash b := \begin{cases} -(|a| \wedge |b|) & \text{if } \operatorname{sign} a \neq \operatorname{sign} b \\ |a| \wedge |b| & \text{else,} \end{cases}$$

and for any sequence $a_1, \ldots, a_n$ in $[-1,1]$, the expression $\langle \bigotimes_{i=1}^{n} a_i \rangle$ is a shorthand for $(\bigotimes_{i=1}^{n} a_i^+) \otimes (-\bigotimes_{i=1}^{n} a_i^-)$.

Then, the Sugeno integral for bi-capacities becomes

$$\mathcal{S}_v(f) = \langle \bigotimes_{i=1}^{n} \Big[|f_{\sigma(i)}| \oslash v(A_{\sigma(i)} \cap N^+, A_{\sigma(i)} \cap N^-)\Big]\rangle. \quad (11)$$

Proposition 5 *Let v be a bi-capacity satisfying $v(A,B) = \nu_1(A) \otimes (-\nu_2(B))$ for all $(A,B) \in \mathcal{Q}(N)$, with ν_1, ν_2 being two normalized capacities on N (v is called a* $\vee$-CPT bi-capacity*). Then the Sugeno integral reduces to*

$$\mathcal{S}_v(f) := \mathcal{S}_{\nu_1}(f^+) \otimes (-\mathcal{S}_{\nu_2}(f^-)).$$

Note that if $\nu_1 = \nu_2$ (v could then be called a $\vee$-*symmetric bi-capacity*), then $\mathcal{S}_v$ is the symmetric Sugeno integral [5].

Proof: Denote by σ a permutation on N such that $|f|$ is non-decreasing. Since f^+, f^-, ν_1, ν_2 are non negative, we have as in the proof of Prop. 4

$$\mathcal{S}_{\nu_1}(f^+) = \bigvee_{i=1}^{n} \Big[f^+_{\sigma(i)} \wedge \nu_1(A_{\sigma(i)} \cap N^+) \Big]$$
$$\mathcal{S}_{\nu_2}(f^-) = \bigvee_{i=1}^{n} \Big[f^-_{\sigma(i)} \wedge \nu_2(A_{\sigma(i)} \cap N^-) \Big].$$

Using the definition of v, we get

$$\mathcal{S}_v(f) = \langle \mathop{\varovee}_{i=1}^{n} \Big[|f_{\sigma(i)}| \owedge [\nu_1(A_{\sigma(i)} \cap N^+) \varovee (-\nu_2(A_{\sigma(i)} \cap N^-))] \Big] \rangle.$$

Due to the definition of $\langle\cdot\rangle$, we have to show that if $\mathcal{S}_{\nu_1}(f^+)$ is larger (resp. smaller) than $\mathcal{S}_{\nu_2}(f^-)$, then the maximum of positive terms is equal to $\mathcal{S}_{\nu_1}(f^+)$ and is larger in absolute value than the maximum of negative terms (resp. the maximum of absolute value of negative terms is equal to $\mathcal{S}_{\nu_2}(f^-)$ and is larger in absolute value than the maximum of positive terms).

Let us consider $\sigma(i) \in N^+$. Two cases can happen.

- if $\nu_1(A_{\sigma(i)} \cap N^+) > \nu_2(A_{\sigma(i)} \cap N^-)$, then the corresponding term reduces to $f^+_{\sigma(i)} \owedge \nu_1(A_{\sigma(i)} \cap N^+)$. This term is identical to the ith term in $\mathcal{S}_{\nu_1}(f^+)$.
- if not, the ith term in $\mathcal{S}_v(f)$ reduces to $-f^+_{\sigma(i)} \owedge \nu_2(A_{\sigma(i)} \cap N^-)$. Due to monotonicity of ν_1, this will be also the case for all subsequent indices $\sigma(i+1), \ldots \sigma(i+k)$, provided they belong to N^+. Moreover, assuming $\sigma(i+k+1) \in N^-$, we have

$$\begin{aligned} |f_{\sigma(i+k+1)}| \owedge \Big[\nu_1(A_{\sigma(i+k+1)} \cap N^+) \varovee (-\nu_2(A_{\sigma(i+k+1)} \cap N^-)) \Big] \\ = -|f_{\sigma(i+k+1)}| \owedge \nu_2(A_{\sigma(i+k+1)} \cap N^-) \\ \leq |f_{\sigma(j)}| \owedge \nu_2(\underbrace{A_{\sigma(j)} \cap N^-}_{A_{\sigma(i+k+1)} \cap N^-}), \quad \forall j = i, \ldots, i+k. \end{aligned}$$

  Hence, in the negative part of $\mathcal{S}_v(f)$, the term in $\sigma(i+k+1)$ remains, while all terms in $\sigma(i), \ldots, \sigma(i+k)$ are cancelled, and it coincides with the term in $\mathcal{S}_{\nu_2}(f^-)$. On the other hand, in $\mathcal{S}_{\nu_1}(f^+) \varovee (-\mathcal{S}_{\nu_2}(f^-))$, the term in $\sigma(i)$ in $\mathcal{S}_{\nu_1}(f^+)$ is smaller than the term in $\sigma(i+k+1)$ of $\mathcal{S}_{\nu_2}(f^-)$, so that the term in $\sigma(i)$ cannot be the result of the computation, and thus it can be discarded from $\mathcal{S}_{\nu_1}(f^+)$.

A similar reasoning can be done with $\sigma(i) \in N^-$. ■

The following result expresses the Choquet integral in terms of the Möbius transform.

Proposition 6 *For any bi-capacity v, any real valued function f on N,*

$$\mathcal{C}_v(f) = \sum_{B\subset N} m(\emptyset, B)\Big(\bigwedge_{i\in B^c\cap N^-} f_i\Big) + \sum_{\substack{(A,B)\in\mathcal{Q}(N)\\ A\neq\emptyset}} m(A,B)\Big[\Big(\bigwedge_{i\in(A\cup B)^c\cap N^-} f_i + \bigwedge_{i\in A} f_i\Big)\vee 0\Big]$$

with the convention $\wedge_\emptyset f_i := 0$.

The proof is based on the following result.

Lemma 1 *For any real valued function f on N, the Choquet integral of f w.r.t. a bi-unanimity game $u_{(A,B)}$ (see (14)) is given by:*

$$\mathcal{C}_{u_{(A,B)}}(f) = \begin{cases} 0, & \text{if } (A,B) = (\emptyset, N) \\ (\bigwedge_{i\in B^c} f_i)\wedge 0, & \text{if } (A,B) = (\emptyset, B) \\ (\bigwedge_{i\in A} f_i)\vee 0, & \text{if } (A,B) = (A, A^c), A\neq\emptyset \\ (\bigwedge_{i\in(A\cup B)^c\cap N^-} f_i + \bigwedge_{i\in A} f_i)\vee 0 & , \text{otherwise.} \end{cases} \tag{12}$$

Proof: Using (10), we get:

$$\mathcal{C}_{u_{(A,B)}}(f) = \sum_{i=1}^{n} |f_{\sigma(i)}|\Big[u_{(A,B)}(A_{\sigma(i)}\cap N^+, A_{\sigma(i)}\cap N^-) - u_{(A,B)}(A_{\sigma(i+1)}\cap N^+, A_{\sigma(i+1)}\cap N^-)\Big]$$

with the above defined notations. For a given $i\in N$, the difference into brackets is non zero only in the following cases: the left term is 0 and the second one is equal to 1 (case 1_i), or the converse (case 2_i). Case 1_i happens if and only if $A_{\sigma(i+1)}\cap N^+ \supset A$ and $A_{\sigma(i+1)}\cap N^- \subset B$, and either $A_{\sigma(i)}\cap N^+ \not\supset A$ or $A_{\sigma(i)}\cap N^- \not\subset B$. Observing that the 3d condition cannot occur if the 1st holds, this amounts to:

$$\text{Case } 1_i \Leftrightarrow \begin{cases} (\alpha_i) & \sigma(i)\in B^c\cap N^- \\ (\beta_i) & A_{\sigma(i)}\cap N^+ \supset A \\ (\gamma_i) & A_{\sigma(i+1)}\cap N^- \subset B. \end{cases}$$

Proceeding similarly with case 2_i, we get:

$$\text{Case } 2_i \Leftrightarrow \begin{cases} (\alpha_i') & \sigma(i)\in A\cap N^+ \\ (\beta_i) & A_{\sigma(i)}\cap N^+ \supset A \\ (\gamma_i) & A_{\sigma(i+1)}\cap N^- \subset B. \end{cases}$$

We remark that only the first conditions differ.

Let us suppose that $(A,B) = (\emptyset, N)$. Then, for any $i\in N$, neither case 1_i nor case 2_i can happen, since conditions (α_i), (α_i') cannot hold. Hence $\mathcal{C}_{u_{(\emptyset,N)}}(f) = 0$. This proves the first line in (12).

Let us suppose $(A,B) = (\emptyset, B)$, $B\neq N$. Then for any $i\in N$, case 2_i can never occur since condition (α_i') is not fulfilled, and condition (β_i) is always fulfilled. If $B^c\cap N^- \neq\emptyset$, there is at least one i such that both (α_i) and (γ_i) are fulfilled, namely $\sigma(i) \in B^c \cap N^-$ such that $|f_{\sigma(i)}|$ is maximum (this forces $\sigma(i+1)$ to be either in N^+ or in B). Hence we get $\mathcal{C}_{u_{(\emptyset,B)}}(f) = \wedge_{i\in B^c} f_i$ under $B^c \cap N^- \neq \emptyset$, and 0 otherwise. This proves the second line of (12).

Let us suppose $(A, B) = (A, A^c)$. If condition β_i holds for some i, then we have $A \subset N^+$. For any $i \in N$, case 1_i cannot occur since $\sigma(i) \in A \cap N^-$ by condition α_i, so that by condition β_i, $\sigma(i) \in N^+$, which contradicts condition α_i. Any $\sigma(i) \in A$ fulfills conditions α'_i and γ_i, while fulfilling also condition β_i imposes to choose $\sigma(i) \in A$ such that $f_{\sigma(i)}$ is minimum. Hence, under these conditions we get $\mathcal{C}_{u_{(A,A^c)}}(f) = \wedge_{i\in A} f_i$. This proves the third line of (12).

Let us consider the general case, with $A \neq \emptyset, B^c$. Let us suppose that case 1_i occurs for some $i \in N$. Then necessarily, $A \subset N^+$ by condition β_i, $\sigma(i) \in (A \cup B)^c \cap N^- \neq \emptyset$ (conditions α_i, β_i), and moreover $\sigma(i)$ is such that $|f_{\sigma(i)}|$ is maximum on $(A \cup B)^c \cap N^-$ and

$$|f_{\sigma(i)}| < \wedge_{j\in A} f_{\sigma(j)} \tag{13}$$

(conditions β_i, γ_i). Under the assumption that case 1_i holds, let us show that case 2_j must occur for some j. Since $\emptyset \neq A \subset N^+$, condition α'_j holds for some j. For any such j, $j > i$ by (13). Now, condition β_j imposes to choose j such that $f_{\sigma(j)}$ is minimum in $A \cap N^+$. j being such defined, let us show that γ_j holds. Suppose it is not the case. This means that there exists j' such that $\sigma(j') \in A_{\sigma(j+1)} \cap N^-$ and $\sigma(j') \notin B$. Since $A \subset N^+$, this means that $\sigma(j') \in (A \cup B)^c \cap N^-$. However, $i < j < j'$, so that $|f_{\sigma(j')}| > |f_{\sigma(i)}|$, which contradicts the definition of $\sigma(i)$. The reverse result can be shown as well, so that either both cases hold or both fail to hold. If both hold, $\mathcal{C}_{u_{(A,B)}}(f) = \wedge_{i\in(A\cup B)^c\cap N^-} f_i + \wedge_{i\in A} f_i$, which is ≥ 0 by (13). This proves the last line of (12). ■

Using the above lemma, the proof of Prop. 6 is immediate considering Eq. (15) and convention $\wedge_\emptyset = 0$.

6 The Shapley value and interaction indices

We consider now bi-capacities as *bi-cooperative games*, as introduced by Bilbao *et al.* [...]. Their definition coincide with our definition of bi-capacities. An example of bi-cooperative game is the one of *ternary voting games* as proposed by Felsenthal and Machover [...], where the value of v is limited to $\{-1, 1\}$. In ternary games, $v(S, T)$ for any $(S, T) \in \mathcal{Q}(N)$ is interpreted as the result of voting (+1: accepted, -1: rejected) when S is the set of voters voting in favor and T the set of voters voting against. $N \setminus S \cup T$ is the set of abstainers. For (general) bi-cooperative games, one can keep the same kind of interpretation: $v(S, T)$ is the worth of the coalition S when T is the opposite coalition, and $N \setminus S \cup T$ is the set of indifferent (indecise) players. [....to be modified....]

An important concept in game theory is the Shapley value [14] and other related indices (e.g. Banzhaf index, probabilistic values), as well as their generalizations as interaction indices.

6.1 Bi-unanimity games

A direct transposition of the notion of unanimity game leads to the following. Let (S, S') in $\mathcal{Q}(N)$. The *bi-unanimity game* centered on (S, S') is defined by:

$$u_{(S,S')}(T, T') = \begin{cases} 1, & \text{if } T \supset S \text{ and } T' \subset S' \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

It is easy to see by (9) that the Möbius transform of $u_{(S,S')}$ is

$$m^{u_{(S,S')}}(T, T') = \begin{cases} 1, & \text{if } (T, T') = (S, S') \\ 0, & \text{otherwise.} \end{cases}$$

Hence, as in the classical case, the set of all bi-unanimity games is a basis for bi-capacities:

$$v(T, T') = \sum_{(S,S') \in \mathcal{Q}(N)} m(S, S') u_{(S,S')}(T, T'). \quad (15)$$

Remark that $u_{(S,S')}$ is not a normalized bi-capacity since $u_{(S,S')}(\emptyset, N) \neq -1$.

6.2 Derivatives of bi-capacities

We extend the notion of derivative of a set function to bi-cooperative games (in fact to any function on $\mathcal{Q}(N)$). As bi-cooperative games are defined on $\mathcal{Q}(N)$, so should be the variables used in derivation. For any $i \in N$, the *left-derivative* with respect to i of v at point (S, T) is given by:

$$\Delta_{i,\emptyset} v(S, T) := v(S \cup i, T) - v(S, T), \quad \forall (S, T) \in \mathcal{Q}(N \setminus i). \quad (16)$$

Similarly, the *right-derivative* is given by:

$$\Delta_{\emptyset,i} v(S, T) := v(S, T) - v(S, T \cup i), \quad \forall (S, T) \in \mathcal{Q}(N \setminus i). \quad (17)$$

The monotonicity of v entails that the left derivative is non negative, while the right derivative is non positive. One can also introduce the *derivative* w.r.t. i by

$$\Delta_i v(S, T) := \Delta_{i,\emptyset} v(S, T) + \Delta_{\emptyset,i} v(S, T) = v(S \cup i, T) - v(S, T \cup i). \quad (18)$$

Left and right derivatives permit to define in general the derivative with respect to any element $(\emptyset, \emptyset) \neq (S, T)$ in $\mathcal{Q}(N)$ by the recursive relation:

$$\Delta_{S,T} v(K, L) := \Delta_{i,\emptyset}(\Delta_{S \setminus i,T} v(K, L)) = \Delta_{\emptyset,i}(\Delta_{S,T \setminus i} v(K, L)), \forall (K, L) \in \mathcal{Q}(N \setminus (S \cup T)). \quad (19)$$

We have for example

$$\Delta_{i,j} v(K, L) = v(K \cup i, L) - v(K \cup i, L \cup j) - v(K, L) + v(K, L \cup j)$$
$$\Delta_{ij,\emptyset} v(K, L) = v(K \cup ij, L) - v(K \cup i, L) - v(K \cup j, L) + v(K, L).$$

The general expression for the (S,T)-derivative is given by:

$$\Delta_{S,T}v(K,L) = \sum_{\substack{S'\subset S\\T'\subset T}} (-1)^{(s-s')-t'} v(K\cup S', L\cup T'), \quad \forall (K,L)\in \mathcal{Q}(N\setminus(S\cup T)). \tag{20}$$

As before, we introduce the derivative w.r.t. S for any $S\subset N$

$$\Delta_S v(K,L) := \Delta_{S,\emptyset}v(K,L) + \Delta_{\emptyset,S}v(K,L).$$

We express the derivative in terms of the Möbius transform. The starting point is the following.

Lemma 2 *For any $i\in N$ and any $(S,T)\in\mathcal{Q}(N\setminus i)$,*

$$\Delta_{i,\emptyset}v(S,T) = \sum_{(S',T')\in[(i,i^c),(S\cup i,T)]} m(S',T') \tag{21}$$

$$\Delta_{\emptyset,i}v(S,T) = \sum_{(S',T')\in[(\emptyset,i^c),(S,T)]} m(S',T') \tag{22}$$

$$\Delta_i v(S,T) = \sum_{(S',T')\in[(\emptyset,i^c),(S\cup i,T)]} m(S',T'). \tag{23}$$

Proof: Let us show (21). For any $(S,T)\in\mathcal{Q}(N\setminus i)$,

$$\begin{aligned}\Delta_{i,\emptyset}v(S,T) =& v(S\cup i,T) - v(S,T)\\ =& \sum_{(S',T')\leq(S\cup i,T)} m(S',T') - \sum_{(S',T')\leq(S,T)} m(S',T')\\ =& \sum_{(S',T')\leq(S,T)} m(S'\cup i,T').\end{aligned}$$

On the other hand,

$$\begin{aligned}[(i,i^c),(S\cup i,T)] =& \{(S',T')\in\mathcal{Q}(N) | i\in S'\subset S\cup i, T\subset T'\subset i^c\}\\ =& \{(S'\cup i,T')\in\mathcal{Q}(N) | S'\subset S, T'\supset T\}\end{aligned}$$

hence the result. Similarly, we have

$$\begin{aligned}\Delta_{\emptyset,i}v(S,T) =& v(S,T) - v(S,T\cup i)\\ =& \sum_{(S',T')\leq(S,T)} m(S',T') - \sum_{(S',T')\leq(S,T\cup i)} m(S',T')\\ =& \sum_{S'\subset S, T'\supset T, i\notin T'} m(S',T') = \sum_{(S',T')\in[(\emptyset,i^c),(S,T)]} m(S',T').\end{aligned}$$

Eq. (23) follows immediately since $[(i,i^c),(S\cup i,T)]\cap[(\emptyset,i^c),(S,T)] = \emptyset$. ■

By induction, one can show the following general result.

Proposition 7 *For any* $(\emptyset,\emptyset)\neq(S,T)$ *in* $\mathcal{Q}(N)$,

$$\Delta_{S,T}v(K,L)=\sum_{(S',T')\in[\bigvee_{i\in S}(i,i^c)\vee\bigvee_{j\in T}(\emptyset,j^c),(S\cup K,L)]}m(S',T'),\quad\forall(K,L)\in\mathcal{Q}(N\setminus(S\cup T))\qquad(24)$$

$$\begin{aligned}\Delta_S v(K,L)&=\sum_{(S',T')\in[\bigvee_{i\in S}(i,i^c),(S\cup K,L)]}m(S',T')+\sum_{(S',T')\in[\bigvee_{j\in S}(\emptyset,j^c),(K,L)]}m(S',T')\\&=\sum_{\substack{C\subset K\\ D\subset N\setminus(S\cup L)}}\Big[m(S\cup C,L\cup D)+m(C,L\cup D)\Big],\quad\forall(K,L)\in\mathcal{Q}(N\setminus S).\qquad(25)\end{aligned}$$

Proof: We prove (24) by induction over (S,T). The result holds for $(i,\emptyset)$ and $(\emptyset,i)$ due to Lemma 2. We suppose that the above formula holds up to a given cardinality of S and T. Let us compute $\Delta_{S\cup k,T}v(K,L)$, for some $k\in N\setminus(S\cup T)$, and any $(K,L)\in\mathcal{Q}(N\setminus(S\cup T\cup k))$. As a preliminary remark, note that

$$\bigvee_{i\in S}(i,i^c)\vee\bigvee_{j\in T}(\emptyset,j^c)=(S,N\setminus(S\cup T)).$$

We have

$$\begin{aligned}\Delta_{S\cup k,T}v(K,L)&=\Delta_{(k,\emptyset)}(\Delta_{S,T}v(K,L))=\Delta_{S,T}v(K\cup k,L)-\Delta_{S,T}v(K,L)\\&=\Big[\sum_{(S',T')\in[(S,N\setminus(S\cup T)),(S\cup K\cup k,L)]}m(S',T')-\sum_{(S',T')\in[(S,N\setminus(S\cup T)),(S\cup K,L)]}m(S',T')\Big]\\&=\Big[\sum_{\substack{S\subset S'\subset S\cup K\cup k\\ L\subset T'\subset N\setminus(S\cup T)}}m(S',T')-\sum_{\substack{S\subset S'\subset S\cup K\\ L\subset T'\subset N\setminus(S\cup T)}}m(S',T')\Big]\\&=\sum_{\substack{S\cup k\subset S'\subset S\cup K\cup k\\ L\subset T'\subset N\setminus(S\cup T\cup k)}}m(S',T')\\&=\sum_{(S',T')\in[(S\cup k,N\setminus(S\cup T\cup k)),(S\cup K\cup k,L)]}m(S',T')\\&=\sum_{(S',T')\in[\bigvee_{i\in S\cup k}(i,i^c)\vee\bigvee_{j\in T}(\emptyset,j^c),(S\cup K\cup k,L)]}m(S',T')\end{aligned}$$

which is the desired result. The case of $\Delta_{S,T\cup k}v(K,L)$ works similarly. Lastly, Eq. (25) comes directly from (24). ■

Remark that for any $(S,T)\in\mathcal{Q}(N)$,

$$\begin{aligned}\Delta_{S,T}v(\emptyset,N\setminus(S\cup T))&=m(S,N\setminus(S\cup T))\\ m(S,T)&=\Delta_{S,N\setminus(T\cup S)}v(\emptyset,T).\end{aligned}$$

6.3 The Shapley value of bi-cooperative games

The Shapley value for bi-cooperative games can be defined axiomatically by introducing axioms which are straightforward extensions of the original axioms, plus an additional symmetry axiom [......detail if necessary.......]. For any player $i \in N$, it is shown in [...] that the Shapley value of i for v is:

$$\phi^v(i) = \sum_{S \subset N \setminus i} \frac{(n-s-1)!s!}{n!} \Big[v(S \cup i, N \setminus (S \cup i)) - v(S, N \setminus S) \Big]. \tag{26}$$

Note that the term into brackets is simply $\Delta_i v(S, N \setminus (S \cup i))$, so that the Shapley value uses the value of v only at vertices.

It is immediate to see that if v is of the CPT type, i.e. $v(S,T) = \nu_1(S) - \nu_2(T)$, then

$$\phi^v(i) = \phi^{\nu_1}(i) + \phi^{\nu_2}(i), \quad \forall i \in N,$$

where $\phi^{\nu_1}, \phi^{\nu_2}$ are the (classical) Shapley values of ν_1 and ν_2. Recalling that for any game ν, $\phi^\nu(i) = \phi^{\overline{\nu}}(i)$ for any $i \in N$, we get $\phi^v(i) = 2\phi^\nu(i)$ for any symmetric or asymmetric game v. If v is an additive bi-capacity, we have $\phi^v(i) = v(i, \emptyset) - v(\emptyset, i)$.

The following expression gives the Shapley value in terms of the Möbius transform.

Proposition 8 *Let v be a bi-cooperative game on N. For any $i \in N$,*

$$\phi^v(i) = \sum_{(S,S') \geq (\emptyset, i^c)} \frac{1}{n-s'} m(S,S').$$

We need the following lemma.

Lemma 3

$$\sum_{i=0}^{k} \frac{(n-i-1)!k!}{n!(k-i)!} = \frac{1}{n-k}.$$

Proof:

$$\sum_{i=0}^{k} \frac{(n-i-1)!k!}{n!(k-i)!} = \frac{1}{n} + \frac{k}{n(n-1)} + \cdots + \frac{k!}{n(n-1)\cdots(n-k)}$$
$$= \frac{(n-1)\cdots(n-k) + k(n-2)\cdots(n-k) + k(k-1)(n-3)\cdots(n-k) + \cdots + k!}{n(n-1)\cdots(n-k)}.$$

It suffices to show that the numerator is $n(n-1)\cdots(n-k+1)$. Summing the last two terms of the numerator, then the last three terms and so on, we get successively:

$$
\begin{aligned}
k(k-1)\cdots 2(n-k)+k! &= k\cdots 2(n-k+1)\\
k\cdots 3(n-k+1)(n-k)+k\cdots 2(n-k+1) &= k\cdots 3(n-k+1)(n-k+2)\\
&\vdots\\
k\cdots i(n-k+i-2)\cdots(n-k)+ &\\
k\cdots(i-1)(n-k+i-2)\cdots(n-k+1) &= k\cdots i(n-k+i-2)\cdots\\
&\quad\cdots(n-k+1)(n-k+i-1)\\
&\vdots\\
k(n-2)\cdots(n-k)+k(k-1)(n-2)\cdots(n-k+1) &= k(n-2)\cdots(n-k+1)(n-1)\\
(n-1)\cdots(n-k)+k(n-1)\cdots(n-k+1) &= (n-1)\cdots(n-k+1)n.
\end{aligned}
$$

■

We can show now Prop. 8.

Proof: We have by Lemma 2

$$\Delta_i v(S, N\setminus(S\cup i)) = \sum_{(S',T')\in[(\emptyset,i^c),(S\cup i,N\setminus(S\cup i))]} m(S',T').$$

Observe that for $S = N\setminus i$, the interval becomes $\uparrow(\emptyset, i^c)$, which contains each interval $[(\emptyset,i^c),(S\cup i, N\setminus(S\cup i))]$ when $S\subset N\setminus i$. Hence,

$$\phi^v(i) = \sum_{(S',T')\in\uparrow(\emptyset,i^c)} m(S',T') \sum_{\substack{S\subset N\setminus i\\ S\cup i\supset S'\\ N\setminus(S\cup i)\subset T'}} \frac{(n-s-1)!s!}{n!}.$$

In the second summation, condition $S\cup i\supset S'$ is redundant. Also, due to the symmetry of the combinatorial factor, it is equivalent to use S or $N\setminus(S\cup i)$ as variable. So the second summation can be rewritten as

$$
\begin{aligned}
\sum_{S\subset T'}\frac{(n-s-1)!s!}{n!} &= \sum_{s=0}^{t'}\frac{(n-s-1)!s!}{n!}\binom{t'}{s}\\
&= \sum_{s=0}^{t'}\frac{(n-s-1)!t'!}{n!(t'-s)!}.
\end{aligned}
$$

Using Lemma 3, the result is proven. ■

6.4 The interaction index

As in [10], the interaction index can be obtained from the Shapley value by a recursion formula. We first introduce necessary notions. Let v be a bi-cooperative game on N, and let

$\emptyset \neq K \subset N$. The *restricted game* $v^{N\setminus K}$ is the game v restricted to players in $N \setminus K$, hence $v^{N\setminus K}(S,T) = v(S,T)$ for any $(S,T) \in \mathcal{Q}(N \setminus K)$, and is not defined outside. The *reduced game* $v^{[K]}$ is the game where all players in K are considered as a single player denoted $[K]$, i.e. the set of players is then $N_{[K]} := (N \setminus K) \cup \{[K]\}$. The reduced game is defined by

$$v^{[K]}(S,T) := v(\phi_{[K]}(S), \phi_{[K]}(T))$$

for any $(S,T) \in \mathcal{Q}(N_{[K]})$, and $\phi_{[K]} : N_{[K]} \longrightarrow N$ is defined by

$$\phi_{[K]}(S) := \begin{cases} S, & \text{if } [K] \notin S \\ (S \setminus [K]) \cup K, & \text{otherwise.} \end{cases}$$

Let us denote by $I^v(S)$ the interaction index for coalition $S \neq \emptyset$ in game v. The recursion formula is [10]

$$I^v(S) = I^{v^{[S]}}([S]) - \sum_{K \subset S, K \neq \emptyset, S} I^{v^{N\setminus K}}(S \setminus K).$$

It can be shown that the interaction index writes

$$I^v(S) = \sum_{T \subset N\setminus S} \frac{(n-s-t)!t!}{(n-s+1)!} \Big[\sum_{L \subset S} (-1)^{s-l} v(L \cup T, N \setminus (T \cup S)) - \sum_{L \subset S} (-1)^{s-l} v(T, N \setminus (L \cup T)) \Big].$$

Observe that this may be written also

$$I^v(S) = \sum_{T \subset N\setminus S} \frac{(n-s-t)!t!}{(n-s+1)!} \Delta_S v(T, N \setminus (S \cup T)).$$

Proposition 9 *Let v be a bi-cooperative game on N. For any $S \subset N$,*

$$I^v(S) = \sum_{(S',T') \geq \bigvee_{i \in S} (i, i^c)} \frac{1}{n - t' - s + 1} m(S',T') + \sum_{(S',T') \in \mathcal{Q}(N\setminus S)} \frac{1}{n - t' - s + 1} m(S',T').$$

Proof: We have by Prop. 7 for any $T \subset N \setminus S$

$$\Delta_S v(T, N\setminus(S \cup T)) = \sum_{(S',T') \in [(S, N\setminus S), (S\cup T, N\setminus(S\cup T))]} m(S',T') + \sum_{(S',T') \in [(\emptyset, N\setminus S), (T, N\setminus(S\cup T))]} m(S',T').$$

Let us study the first term. Observe that for $T = N \setminus S$, the interval becomes $\uparrow (S, N \setminus S)$, which contains all intervals $[(S, N \setminus S), (S \cup T, N \setminus (S \cup T)]$. Hence

$$\sum_{T \subset N\setminus S} \frac{(n-s-t)!t!}{(n-s+1)!} \sum_{(S',T') \in [(S, N\setminus S), (S\cup T, N\setminus(S\cup T))]} m(S',T') = \sum_{(S',T') \in \uparrow(S, N\setminus S)} m(S',T') \sum_{\substack{T \subset N\setminus S \\ S \cup T \supset S' \\ N\setminus(S\cup T) \subset T'}} \frac{(n-s-t)!t!}{(n-s+1)!}.$$

Observe that in the second summation, condition $S \cup T \supset S'$ on T is redundant. Also, due to the symmetry of the combinatorial factor, we can interchange T with $N \setminus (S \cup T)$. So the second summation can be rewritten as

$$\sum_{T \subset T'} \frac{(n-s-t)!t!}{(n-s+1)!} = \sum_{t=0}^{t'} \binom{t'}{t} \frac{(n-s-t)!t!}{(n-s+1)!}$$
$$= \sum_{t=0}^{t'} \frac{(n-s-t)!t'!}{(t'-t)!(n-s+1)!}$$
$$= \frac{1}{n-s-t'+1}$$

by using Lemma 3.

In the second term, for $T = N \setminus S$, the interval becomes $[(\emptyset, N \setminus S), (N \setminus S, \emptyset)] = \mathcal{Q}(N \setminus S)$. The rest works similarly. ■

> NOTE: the above proof will be subsumed by the proof for bi-interaction indices, more clearly written.

This expression shows that if v is k-additive, then $I^v(S) = 0$ for any S of more than k elements, and for any S of exactly k elements, $I^v(S) = m(S, N \setminus S) + m(\emptyset, N \setminus S)$.

Proposition 10 *If v is of the CPT type, with $v(S,T) = \nu_1(S) - \nu_2(T)$, then*

$$I^v(S) = I^{\nu_1}(S) + I^{\overline{\nu_2}}(S),$$

where I^{ν_i} is the (classical) interaction index of ν_i. Since $I^{\overline{\nu_i}}(S) = (-1)^{s+1} I^{\nu_i}(S)$, we have $I^v(S) = 0$ when s is even and v is asymmetric.

Proof: Using Prop. 2 and expression of classical interaction w.r.t. the Möbius transforms of ν_1 and ν_2, we have

$$I^v(S) = \sum_{S' \supset S} \frac{1}{s'-s+1} m(S', N \setminus S') + \sum_{T' \subset N \setminus S} \frac{1}{n-t'-s+1} m(\emptyset, T')$$
$$= \sum_{S' \supset S} \frac{1}{s'-s+1} m^{\nu_1}(S') + \sum_{T' \subset N \setminus S} \frac{1}{n-t'-s+1} m^{\overline{\nu_2}}(N \setminus T')$$
$$= I^{\nu_1}(S) + \sum_{T'' \subset N \setminus S} \frac{1}{t''-s+1} m^{\overline{\nu_2}}(T'')$$
$$= I^{\nu_1}(S) + I^{\overline{\nu_2}}(S).$$

■

6.5 The bi-interaction index

Since bi-cooperative games are defined on $\mathcal{Q}(N)$, the interaction index should be defined for all coalitions in $\mathcal{Q}(N)$. The most natural definition seems to use the derivative, as for the classical case. We propose the following

Definition 9 *Let* $(S,T) \in \mathcal{Q}(N)$. *The* bi-interaction index *w.r.t* (S,T) *is defined by:*

$$I^v(S,T) := \sum_{K \subset N \setminus (S \cup T)} \frac{(n-s-t-k)!k!}{(n-s-t+1)!} \Delta_{S,T} v(K, N \setminus (K \cup S \cup T)).$$

A first observation is that for any $S \subset N$

$$I^v(S) = I^v(S,\emptyset) + I^v(\emptyset, S).$$

Indeed,

$$I^v(S,\emptyset) = \sum_{K \subset N \setminus S} \frac{(n-s-k)!k!}{(n-s+1)!} \Delta_{S,\emptyset} v(K, N \setminus (K \cup S))$$

$$I^v(\emptyset,S) = \sum_{K \subset N \setminus S} \frac{(n-s-k)!k!}{(n-s+1)!} \Delta_{\emptyset,S} v(K, N \setminus (K \cup S))$$

and since $\Delta_S v = \Delta_{S,\emptyset} v + \Delta_{\emptyset,S} v$, the result holds.

This gives immediately

$$\sum_{i \in N} I^v(i,\emptyset) - \sum_{i \in N} I^v(\emptyset, i) = 2 \tag{27}$$

since $\sum_{i \in N} \phi^v(i) = v(N,\emptyset) - v(\emptyset, N)$.

Proposition 11 *Let v be a bi-cooperative game on N. For any $(S,T) \subset \mathcal{Q}(N)$,*

$$\begin{aligned} I^v(S,T) &= \sum_{(S',T') \in \uparrow (\bigvee_{i \in S} (i, i^c) \vee \bigvee_{j \in T} (\emptyset, j^c)) \cap \mathcal{Q}(N \setminus T)} \frac{1}{n-s-t-t'+1} m(S',T') \\ &= \sum_{(S',T') \in [(S, N \setminus (S \cup T)), (N \setminus T, \emptyset)]} \frac{1}{n-s-t-t'+1} m(S',T'). \end{aligned}$$

Proof: By Prop. 7, we have

$$\Delta_{S,T} v(K, N \setminus (K \cup S \cup T)) = \sum_{(S'T') \in \underbrace{[(S, N \setminus (S \cup T)), (S \cup K, N \setminus (K \cup S \cup T))]}_{\substack{S \subset S' \subset S \cup K \\ N \setminus (K \cup S \cup T) \subset T' \subset N \setminus (S \cup T) \\ S' \cap T' = \emptyset}}} m(S',T').$$

When $K = N \setminus (S \cup T)$, the interval becomes $[(S, N \setminus (S \cup T)), (N \setminus T, \emptyset)]$, or equivalently $\uparrow (\bigvee_{i \in S} (i, i^c) \vee \bigvee_{j \in T} (\emptyset, j^c)) \cap \mathcal{Q}(N \setminus T)$. This interval contains all intervals $[(S, N \setminus (S \cup T)), (S \cup K, N \setminus (K \cup S \cup T))]$ since $S \cup K \subset N \setminus T$. Hence,

$$\sum_{K \subset N \setminus (S \cup T)} \frac{(n-s-t-k)!k!}{(n-s-t+1)!} \Delta_{S,T} v(K, N \setminus (K \cup S \cup T))$$
$$= \sum_{(S',T') \in [(S, N \setminus (S \cup T)), (N \setminus T, \emptyset)]} m(S', T') \sum_{\substack{K \subset N \setminus (S \cup T) \\ S \cup K \supset S' \\ N \setminus (K \cup S \cup T) \subset T'}} \frac{(n-s-t-k)!k!}{(n-s-t+1)!}.$$

Observe that in the second summation, condition $S \cup K \supset S'$ is redundant. Indeed, we have $N \setminus (K \cup S \cup T) \subset T' \Leftrightarrow K \cup S \cup T \supset N \setminus T'$. Since $N \setminus T' \supset S'$ and $T \cap S' = \emptyset$, we deduce $S \cup K \supset S'$.

Using this fact and letting $K' := N \setminus (K \cup S \cup T)$, the second summation becomes:

$$\begin{aligned}
\sum_{N \setminus (K \cup S \cup T) \subset T'} \frac{(n-s-t-k)!k!}{(n-s-t+1)!} &= \sum_{K' \subset T'} \frac{k'!(n-k'-s-t)!}{(n-s-t+1)!} \\
&= \sum_{k'=0}^{t'} \binom{t'}{k'} \frac{k'!(n-k'-s-t)!}{(n-s-t+1)!} \\
&= \sum_{k'=0}^{t'} \frac{t'!(n-k'-s-t)!}{(n-s-t+1)!(t'-k')!} \\
&= \frac{1}{n-s-t-t'+1}
\end{aligned}$$

using Lemma 3. ■

The above proposition contains Prop. 9 and Prop. 8, as it can be checked.

We examine the case of k-additive bi-capacities and CPT-type bi-capacities.

Proposition 12 *(i) If v is a k-additive bi-capacity, then*

$$I^v(S,T) = 0, \quad \forall (S,T) \in \mathcal{Q}(N) \text{ such that } |S \cup T| > k \tag{28}$$
$$I^v(S,T) = m(S, N \setminus (S \cup T)), \quad \forall (S,T) \in \mathcal{Q}(N) \text{ such that } |S \cup T| = k. \tag{29}$$

(ii) If v is of CPT type, then $I^v_{S,T} = 0$ unless $S = \emptyset$ or $T = \emptyset$.

Proof: (i) v is k-additive iff $m(S', T') = 0$ for all T' such that $t' < n - k$. Using Prop. 11, we see that in the summation, $T' \subset N \setminus (S \cup T)$. Consequently, if $|S \cup T| > k$, $m(S', T')$ will be always 0, and so $I^v(S,T) = 0$.

Now, if $|S \cup T| = k$, only $T' = N \setminus (S \cup T)$ gives a non zero term. For any T', we have $S' \subset N \setminus T'$. Since we have also the condition $S \subset S' \subset N \setminus T$, the only solution is $S' = S$, hence the result.

(ii) By Prop. 2, we know that $m(S', T') \neq 0$ iff $S' = \emptyset$ or $S' = N \setminus T'$. In the expression of $I^v(S,T)$ of Prop. 11, the first condition implies $S = \emptyset$, while the second implies $T = \emptyset$. ■

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